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INDEX.

PAGE

Basset, A. B. On the Steady Motion of an Annular Mass of Rotating Liquid, . . 172

Bolza, Oskar. On the Construction of Intransitive Groups, ... 195

Cayley, A. On the Surfaces with Plane or Spherical Curves of Curvature, . . 71, 293

On the Theory of Groups, ... 139

Fields, J. C. The Expression of any Differential Coefficient of a Function of a Function of any number of Variables by aid of the corresponding Differential Coefficients of any *n* Powers of the Function, where *n* is the Order of the Differential Coefficient, ... 388

Fine, Henry B. On the Functions defined by Differential Equations, with an Extension of the Puiseux Polygon Construction to these Equations, ... 317

Franklin, F. Note on the Double Periodicity of the Elliptic Functions, . . . 283

Goursat, E. Sur les solutions singulières des équations différentielles simultanées, . 329

Heun, Karl. Die Herstellung einer lineären Differentialgleichung aus einem gegebenen Element der Integralfunction, ... 215

Johnson, William Woolsey. On the Integrals in Series of Binomial Differential Equations, ... 37

Koenigsberger, Leo. Über die Reduction von Integralen transcendenter Functionen, ... 221

Lie, Sophus. Die Begriffe Gruppe und Invariante, ... 182

Love, A. E. H. Vortex Motion in certain Triangles, ... 158

MacMahon, Capt. P. A. Memoir on a New Theory of Symmetric Functions, . . 1

Morley, F. On the Geometry of a Nodal Circular Cubic, ... 307

d'Ocagne, Maurice. Sur certaines courbes qu'on peut adjoindre aux courbes planes pour l'étude de leurs propriétés infinitésimales, ... 55

Perott, Joseph. Remarque au sujet du théorème d'Euclide sur l'infinité du nombre des nombres premiers, ... 99

Picard, Emile. Sur les formes quadratiques binaires à indéterminées conjuguées et les fonctions fuchsiennes, ... 187

Rowland, Henry A. Electromagnetic Waves and Oscillations at the Surface of Conductors, ... 373

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Memoir on a New Theory [illegible]

By Captain P. A. MacMah[illegible]

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In what follows, I [illegible] of an arbitrary partition [illegible] theory as the particular [illegible] weight) which is composed of [illegible]

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Memoir on a New Theory of Symmetric Functions.

By Captain P. A. MacMahon, R. A., *Woolwich, England.*

In a communication recently made to the London Mathematical Society, I have sketched out an extension of the algebra of the theory of symmetrical functions and have established the bases of a wide development.

For the sake of unity, as well as for the convenience of the readers of this journal, I repeat some definitions and preliminary theorems which are of great moment to the due comprehension of what follows.

The main object of this memoir is to show clearly the proper place of the "Symmetric Function Tables" as studied by Hirsch, Cayley, Durfee and other mathematicians in Europe and America, in the algebra of such functions; to point out that the fact of their existence depends upon a wide theorem of algebraic reciprocity which leads to an equally wide theorem of algebraic expressibility, and that they are a particular case, and not the most important case from the point of view of application, of a system of such tables.

I indicate an application to the general theory of binary forms, which as regards ground forms and syzygies is of considerable promise.

It has been usual to discuss and develop the theory by reference to the weight of the involved symmetric functions; tables have thus been constructed of weights 1, 2, 3, ... , and laws and formulae appertaining thereto have been evolved; some of these laws and formulae are in regard to an arbitrary weight, but so far as my knowledge extends no attempt has hitherto been made to develop the theory from a more general standpoint.

In what follows, I regard the whole theory as arising from the discussion of an arbitrary partition of an arbitrary weight, and bring out the ordinary theory as the particular case corresponding to that partition of w (the arbitrary weight) which is composed of w units.

The relation of the ordinary to the present inclusive theory will become clear as the investigation proceeds, and it will in particular be interesting to note those theorems which are perfectly general for every partition, and those which are peculiar to single partitions.

Definitions.

A number is partitioned into parts by writing down a set of positive numbers which result in the number when added together; each of the constituent numbers is a part of the partition, and the parts are usually written in descending order from left to right and enclosed in a bracket ( ).

A partition is separated into separates by writing down a set of partitions, each separate partition in its own brackets, from left to right, so that when all the parts of these partitions are assembled in a single bracket, the partition which is separated is reproduced. Thus of a partition

$$(p_1p_2p_3p_4p_5),$$

separations are

$$(p_1p_2)(p_3p_4)(p_5),$$
$$(p_1p_2p_3)(p_4p_5).$$

Professor Sylvester has termed a number, quâ its partitions, the partible number; so here we may term a partition, quâ its separations, the separable partition. It is convenient to order the separates of a separation from left to right, in descending order as regards weight.

If the successive weights of the separates be

$$w_1, w_2, w_3, \ldots,$$

I speak of a separation of species partition

$$(w_1w_2w_3 \ldots).$$

The sum of the highest parts of the several separates I further call the "degree" of the separation. The characteristics of a separation are

(i) the weight,
(ii) the separable partition,
(iii) the species partition,
(iv) the degree;

to which may be added

(v) the multiplicity,

where, if the separation be

$$(p_1 \dots .)^{j_1}(p_s \dots .)^{j_2}(p_t \dots .)^{j_3} \dots .,$$

the multiplicity is defined by the succession of indices

$$j_1 j_2 j_3 \dots .$$

Theorem of Reciprocity.

Partitions being symbolical representations of symmetric functions, let

$$X_1 = (1)\,x_1,$$
$$X_2 = (2)\,x_2 + (1^2)\,x_1^2,$$
$$X_3 = (3)\,x_3 + (21)\,x_2x_1 + (1^3)\,x_1^3,$$
$$X_4 = (4)\,x_4 + (31)\,x_3x_1 + (2^2)\,x_2^2 + (21^2)\,x_2x_1^2 + (1^4)\,x_1^4,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$X_m = \sum (m_1m_2m_3 \dots .)\,x_{m_1}x_{m_2}x_{m_3} \dots .,$$

the summation being in regard to every partition

$$(m_1m_2m_3 \dots .)$$

of the number m.

Form the product $$X_{p_1}^{\pi_1}X_{p_2}^{\pi_2}X_{p_3}^{\pi_3} \dots .$$

and observe that, on performing the multiplication, the coefficient of the x term

$$x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3} \dots .$$

is a sum of compound symmetric functions, each of which, to a numerical factor près, is represented symbolically by a separation of the partition

$$(s_1^{\sigma_1}s_2^{\sigma_2}s_3^{\sigma_3} \dots .),$$

and further, that each such separation has a species partition

$$(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3} \dots .).$$

Write then $$X_{p_1}^{\pi_1}X_{p_2}^{\pi_2}X_{p_3}^{\pi_3} \dots . = \sum P x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3} \dots . ;$$

P, being a sum of compound symmetric functions, can be expanded in a sum of monomial symmetric functions. Thus, suppose

$$P = \sum \theta\,(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3} \dots .),$$

we may write

$$X_{p_1}^{\pi_1}X_{p_2}^{\pi_2}X_{p_3}^{\pi_3} \dots . = \sum \sum \theta\,(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3} \dots .)\,x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3} \dots .$$

I have established, in a practically instantaneous manner (loc. cit.), by considering a certain restricted distribution of objects amongst parcels, that in this result the coefficient θ remains unaltered if we interchange the species partition

$$(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots\ldots)$$

and the partition $(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\ldots\ldots)$;

so that we have also

$$X_{\lambda_1}^{l_1}X_{\lambda_2}^{l_2}X_{\lambda_3}^{l_3}\ldots\ldots=\sum\sum\theta\,(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots\ldots)\,x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots\ldots;$$

in other words, the theorem of reciprocity states that the coefficient of

$$(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\ldots\ldots)\,x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots\ldots$$

in the development of $X_{p_1}^{\pi_1}X_{p_2}^{\pi_2}X_{p_3}^{\pi_3}\ldots\ldots$

is equal to the coefficient of

$$(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots\ldots)\,x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots\ldots$$

in the development of $X_{\lambda_1}^{l_1}X_{\lambda_2}^{l_2}X_{\lambda_3}^{l_3}\ldots\ldots$

Observe that the Cayley-Betti law of symmetry connected with ordinary tables of symmetric functions is obtained from this theorem by merely attending to the powers of x_1.

Formation of New Tables.

Writing as before

$$X_{p_1}^{\pi_1}X_{p_2}^{\pi_2}X_{p_3}^{\pi_3}\ldots\ldots=\sum P x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots\ldots,$$

$$P=\sum\theta\,(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3})\ldots\ldots,$$

P is an aggregate of separations of

$$(s_1^{\sigma_1}s_2^{\sigma_2}s_3^{\sigma_3}\ldots\ldots)$$

of species partition $(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots\ldots)$.

Of any separable partition $(s_1^{\sigma_1}s_2^{\sigma_2}s_3^{\sigma_3}\ldots\ldots)$

there is a definite number of species partitions, which is in general less than the number of separations. Forming a product

$$X_{p_1}^{\pi_1}X_{p_2}^{\pi_2}X_{p_3}^{\pi_3}\ldots\ldots$$

for each such species partition, we get an equal number of expressions P, each of which is an aggregate of separations of the corresponding species; on expanding all these expressions P in a series of monomials, we obtain a certain number of different symmetric functions

$$(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\ldots\ldots),$$

each of which, by the law of reciprocity, is a species partition of the separable partition $(s_1^{\sigma_1}s_2^{\sigma_2}s_3^{\sigma_3}\ldots\ldots)$;
hence if there be altogether k species partitions of the separable partition

$$(s_1^{\sigma_1}s_2^{\sigma_2}s_3^{\sigma_3}\ldots\ldots),$$

the development of the corresponding k products

$$X_{p_1}^{\pi_1}X_{p_2}^{\pi_2}X_{p_3}^{\pi_3}\ldots\ldots$$

will, through the k expressions P, lead to precisely k different monomial symmetric functions $(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\ldots\ldots)$
symbolized by identically the same partitions as the species partitions; hence we see directly that, given a separable partition

$$(s_1^{\sigma_1}s_2^{\sigma_2}s_3^{\sigma_3}\ldots\ldots),$$

the species partitions $(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots\ldots)$
are, in some order, the same as the partitions

$$(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\ldots\ldots),$$

and that writing the expressions

$$P$$

in any order vertically and the corresponding species partitions in the same order from left to right, we are able to express the separations

$$P$$

in terms of monomial symmetric functions by means of a table possessing the same row and column symmetry as given by the Cayley-Betti law in the particular case of existing tables.

Theorem: Of a separable partition

$$(s_1^{\sigma_1}s_2^{\sigma_2}s_3^{\sigma_3}\ldots\ldots)$$

the separations P
are expressible by means of monomial symmetric functions, symbolized by partitions which are identical with the species partitions, and a symmetrical table may be thus formed.

By solving a set of simultaneous linear equations, we may now express the monomial symmetric functions which are symbolized by the species partitions of the separations of $(s_1^{\sigma_1}s_2^{\sigma_2}s_3^{\sigma_3}\ldots.)$
in terms of the separations P
which arise from the X-products.

We must also thus get a table possessing the same law of symmetry, as may be easily gathered from the elementary properties of determinants.

Theorem: The monomial symmetric functions, symbolized by the species partitions of the separations of

$$(s_1^{\sigma_1}s_2^{\sigma_2}s_3^{\sigma_3}\ldots.)$$

are expressible in terms of compound symmetric functions, symbolized by the separations P
which arise from the X-products corresponding to the several species partitions, and a table thus formed will possess row and column symmetry.

This last theorem is in fact a law of algebraic expressibility which may be further enunciated as follows:

Theorem of expressibility: If a symmetric function be symbolized by

$$(\lambda\mu\nu\ldots.)$$

and $(\lambda_1\lambda_2\lambda_3\ldots.)$ be any partition of λ,
$(\mu_1\mu_2\mu_3\ldots.)$ " " " μ,
$(\nu_1\nu_2\nu_3\ldots.)$ " " " ν,
. . . .

the symmetric function $(\lambda\mu\nu\ldots.)$
is expressible by means of separations of

$$(\lambda_1\lambda_2\lambda_3\ldots.\mu_1\mu_2\mu_3\ldots.\nu_1\nu_2\nu_3\ldots.).$$

In the ordinary theory, we have the particular case that every symmetric function whatever is expressible by means of the elementary symmetric functions (that is, by the coefficients of the equation, the symmetric functions of whose roots we are considering); observe that these coefficients have partitions composed wholly of units, and that the theorem necessarily arises because *every* member is expressible as a sum of units; the theorem in this case is also one of reducibility, but in general expressibility is not coincident with reducibility, as will appear later on.

I now proceed to show the practical method of constructing symmetrical symmetric function tables corresponding to any partition of any number. For clearness I will take a simple case, viz the partition

$$21^3$$

of the number 5.

We are concerned with symmetric functions of weight 5. Write down the separations of 21^3, viz:

$$\begin{array}{rlll}
(21^3) & \text{of species partition} & & (5), \\
(21^2)(1) & \text{“} & \text{“} & (41), \\
\left.\begin{array}{r}(21)(1^2) \\ (1^3)(2)\end{array}\right\} & \text{“} & \text{“} & (32), \\
(21)(1)^2 & \text{“} & \text{“} & (31^2), \\
(2)(1^2)(1) & \text{“} & \text{“} & (2^21), \\
(2)(1)^3 & \text{“} & \text{“} & (21^3).
\end{array}$$

To each of these separations must be attached the numerical coefficient which arises from the X-product corresponding to the species partition. Thus to find the coefficient of the separation

$$(2)(1^2)(1)$$

we form the product

$$X_2^2X_1 \equiv \{(2)x_2 + (1^2)x_1^2\}^2\{(1)x_1\},$$

because (2^21) is the species partition of

$$(2)(1^2)(1),$$

and pick out the coefficient of

$$(2)(1^2)(1)\,x_2x_1^3$$

in its development. Since

$$X_2^2X_1 = \ldots . + 2\,(2)(1^2)(1)\,x_2x_1^3 + \ldots . ,$$

the required coefficient is $\qquad 2.$

In this manner the proper coefficient of each separation has to be determined, and they may then be written down in a vertical column according to the dictionary, alphabetical or Durfee-order of their species partitions as may seem appropriate to the purpose in hand.

The species partitions themselves are then written in the *same order* from left to right and the skeleton table then appears in the form:

	(5)	(41)	(32)	(31^2)	(2^21)	(21^3)
(21^3)						
$(21^2)(1)$						
$(21)(1^2)+(1^3)(2)$						
$(21)(1)^2$						
$2(2)(1^2)(1)$						
$(2)(1)^3$						

a table which, when filled in, will express the separations in terms of the monomial symmetric functions in a symmetrical manner.

The companion table will express inversely the monomial symmetric functions in terms of the separations, and will have the skeleton form:

	(21^3)	$(21^2)(1)$	$(21)(1^2)+(1^3)(2)$	$(21)(1)^2$	$2(2)(1^2)(1)$	$(2)(1)^3$
(5)						
(41)						
(32)						
(31^2)						
(2^21)						
(21^3)						

This example will, I think, make the method in general clear; we can readily obtain the coefficient to be used with any particular separation by considering the corresponding X-product. Thus, suppose the separation to be

$$(a_1)^{\alpha_1}(a_2)^{\alpha_2}\dots\dots(b_1)^{\beta_1}(b_2)^{\beta_2}\dots\dots(c_1)^{\gamma_1}(c_2)^{\gamma_2}\dots\dots,$$

wherein $a_1, a_2, \dots\dots$ are partitions each of weight a,
$b_1, b_2, \dots\dots$ " " " b,
$c_1, c_2, \dots\dots$ " " " c,
. . . .

then we have to put the numerical coefficient of

$$(a_1)^{\alpha_1}(a_2)^{\alpha_2}\ldots\ldots(b_1)^{\beta_1}(b_2)^{\beta_2}\ldots\ldots(c_1)^{\gamma_1}(c_2)^{\gamma_2}\ldots\ldots$$

in the development of the product

$$X_a^{\alpha_1+\alpha_2+\ldots\ldots}X_b^{\beta_1+\beta_2+\ldots\ldots}X_c^{\gamma_1+\gamma_2+\ldots\ldots}\ldots\ldots,$$

which is $$\frac{(\alpha_1+\alpha_2+\ldots\ldots)!\ (\beta_1+\beta_2+\ldots\ldots)!\ (\gamma_1+\gamma_2+\ldots\ldots)!\ldots\ldots}{\alpha_1!\,\alpha_2!\ldots\ldots\quad\beta_1!\,\beta_2!\ldots\ldots\quad\gamma_1!\,\gamma_2!\ldots\ldots}.$$

The process just laid down for the formation of symmetrical tables is that which would be adopted *a priori* as a result of the general law of reciprocity; there is, however, a practical convenience in modifying the process, in a manner which in no way interferes with the symmetrical character of the tables, so as either to lessen the magnitude of the table numbers or to get rid of fractions.

In the tables which express the separations in terms of the monomials, no fractions can possibly occur, but the numbers may be reduced in magnitude by a modification now to be explained.

In the table which expresses the separations of (21^3), the skeleton of which is given above, the symmetry will remain unchanged if we simultaneously write

$$(2)(1^2)(1) \text{ for } 2(2)(1^2)(1)$$

in the vertical column, and

$$2(2^21) \text{ for } (2^21)$$

in the horizontal row. The effect of this is to diminish the numbers without introducing fractions. To see the reason of this, observe that the coefficient of

$$(2^21)$$

in the development of every separation is necessarily of the form

$$0 \bmod 2$$

(vide Cayley, *Amer. Jour. of Math.*, Vol. VII, p. 59), so that in the original form of table, the vertical column headed (2^21) is necessarily divisible by 2; further, the horizontal row $2(2)(1^2)(1)$ is obviously simultaneously divisible by 2 also.

These divisions will be accomplished, as stated above, by simultaneously writing $(2)(1^2)(1)$ for $2(2)(1^2)(1)$
and $2(2^21)$ for (2^21).

Generally, if in the vertical column any separation occur singly with a coefficient p, the coefficient of the corresponding species partition in the development of every separation of the separable partition is of the form

$$0 \bmod p$$

(vide Cayley loc. cit.); we can thus divide the separation row and the corresponding species partition column simultaneously by p, and this clearly does not interfere with the symmetry. Suppose now that we have in the vertical column of separations an aggregate of separations of a certain species partition, each separation having attached to it its proper coefficient as obtained from the X-product; of these separations, a certain one may be termed the leading separation. To make clear what I mean, I must explain what I wish to be understood by the "leading term" in the development of a separation. Taking any separation, say

$$(51)(432)(1^5),$$

write down the separates in a column, thus

$$\begin{array}{r} 51 \\ 432 \\ 11111 \\ \hline 10,5311 \end{array}$$

and add up the parts vertically, thus obtaining the monomial

$$(10,531^3).$$

I call this term $(10,531^3)$ the leading term in the development of the separation

$$(51)(432)(1^5),$$

and observe particularly that this term, in the development, of necessity occurs with a coefficient unity and precedes in dictionary order any other term arising in the development.

On this understanding I call the leading separation of an aggregate of separations of the same species partitions, that separation whose leading term precedes in dictionary order the leading terms of all other separations in the aggregate. (As to "dictionary order," vide Durfee, *Amer. Jour. of Math.*, Vol. V, p. 349). The rule for modifying the process is to divide the horizontal row and corresponding vertical column by the coefficient of the leading separation. Observe that we thus obtain each leading term with coefficient unity, in itself a considerable advantage.

The inverse tables are further modified by multiplication throughout by some number which will get rid of fractions.

It will be convenient at this point to have before us the complete tables for the first six weights.

Tables of Symmetric Functions.

WEIGHT 1. PARTITION (1).

	(1)
(1)	1

WEIGHT 2. PARTITION (1^2).

	(2)	(1^2)
(1^2)		1
$(1)^2$	1	2

	(1^2)	$(1)^2$
(2)	$\bar{2}$	1
(1^2)	1	

WEIGHT 2. PARTITION (2),

	(2)
(2)	1

WEIGHT 3. PARTITION (1^3).

	(3)	(21)	(1^3)
(1^3)			1
$(1^2)(1)$		1	3
$(1)^3$	1	3	6

	(1^3)	$(1^2)(1)$	$(1)^3$
(3)	3	$\bar{3}$	1
(21)	$\bar{3}$	1	
(1^3)	1		

WEIGHT 3. PARTITION (21).

	(3)	(21)
(21)		1
$(2)(1)$	1	1

	(21)	$(2)(1)$
(3)	$\bar{1}$	1
(21)	1	

WEIGHT 3. PARTITION (3).

	(3)
(3)	1

Weight 4. Partition (1^4).

	(4)	(31)	(2^2)	(21^2)	(1^4)
(1^4)					1
$(1^3)(1)$				1	4
$(1^2)^2$			1	2	6
$(1^2)(1)^2$		1	2	5	12
$(1)^4$	1	4	6	12	24

	(1^4)	$(1^3)(1)$	$(1^2)^2$	$(1^2)(1)^2$	$(1)^4$
(4)	$\overline{4}$	4	2	$\overline{4}$	1
(31)	4	$\overline{1}$	$\overline{2}$	1	
(2^2)	2	$\overline{2}$	1		
(21^2)	$\overline{4}$	1			
(1^4)	1				

Weight 4. Partition (21^2).

	(4)	(31)	$2(2^2)$	(21^2)
(21^2)				1
$(21)(1)$		1	1	2
$(2)(1^2)$		1		1
$(2)(1)^2$	1	2	1	2

	(21^2)	$(21)(1)$	$(2)(1^2)$	$(2)(1)^2$
(4)	1	$\overline{1}$	$\overline{1}$	1
(31)	$\overline{1}$		1	
$2(2^2)$	$\overline{1}$	1	$\overline{1}$	
(21^2)	1			

Weight 4. Partition (2^2).

	(4)	(2^2)
(2^2)		1
$(2)^2$	1	2

	(2^2)	$(2)^2$
(4)	$\overline{2}$	1
(2^2)	1	

Weight 4. Partition (31).

	(4)	(31)
(31)		1
$(3)(1)$	1	1

	(31)	$(3)(1)$
(4)	$\overline{1}$	1
(31)	1	

Weight 4. Partition (4).

	(4)
(4)	1

Weight 5. Partition (1^5).

	(5)	(41)	(32)	(31^2)	(2^21)	(21^3)	(1^5)
(1^5)							1
$(1^4)(1)$						1	5
$(1^3)(1^2)$					1	3	10
$(1^3)(1)^2$				1	2	7	20
$(1^2)^2(1)$			1	2	5	13	30
$(1^2)(1)^3$		1	3	7	12	27	60
$(1)^5$	1	5	10	20	30	60	120

	(1^5)	$(1^4)(1)$	$(1^3)(1^2)$	$(1^3)(1)^2$	$(1^2)^2(1)$	$(1^2)(1)^3$	$(1)^5$
(5)	5	$\overline{5}$	$\overline{5}$	5	5	$\overline{5}$	1
(41)	$\overline{5}$	1	5	$\overline{1}$	$\overline{3}$	1	
(32)	$\overline{5}$	5	$\overline{1}$	$\overline{2}$	1		
(31^2)	5	$\overline{1}$	$\overline{2}$	1			
(2^21)	5	$\overline{3}$	1				
(21^3)	$\overline{5}$	1					
(1^5)	1						

Weight 5. Partition (21^3).

	(5)	(41)	(32)	(31^2)	$2(2^21)$	(21^3)
(21^3)						1
$(21^2)(1)$				1	1	3
$(21)(1^2)+(1^3)(2)$			1	3	1	4
$(21)(1)^2$		1	3	4	3	6
$(2)(1^2)(1)$		1	1	3	1	3
$(2)(1)^3$	1	3	4	6	3	6

	(21^3)	$(21^2)(1)$	$(21)(1^2)+(1^3)(2) \div 5$	$(21)(1)^2$	$(2)(1^2)(1)$	$(2)(1)^3$
(5)	$\overline{5}$	5	5	$\overline{5}$	$\overline{10}$	5
(41)	5		$\overline{5}$		5	
(32)	5	$\overline{5}$	1	2	$\overline{2}$	
(31^2)	$\overline{5}$		2	$\overline{1}$	1	
$2(2^21)$	$\overline{10}$	5	$\overline{2}$	1	$\overline{1}$	
(21^3)	5					

Weight 5. Partition (2^21).

	(5)	(41)	(32)	(2^21)
(2^21)				1
$(2^2)(1)$			1	2
$(21)(2)$		1	1	2
$(2)^2(1)$	1	1	2	2

	(2^21)	$(2^2)(1)$	$(21)(2)$	$(2)^2(1)$
(5)	1	$\overline{1}$	$\overline{1}$	1
(41)	$\overline{1}$	$\overline{1}$	1	
(32)	$\overline{1}$	1		
(2^21)	1			

WEIGHT 5. PARTITION (31^2).

	(5)	(41)	(32)	(31^2)
$(31)^2$				1
(31)(1)		1	1	2
(3)(1^2)		1		1
(3)(1^2)	1	2	1	2

	(31^2)	(31)(1)	(3)(1^2)	(3)$(1)^2$
(5)	1	$\bar{1}$	$\bar{1}$	1
(41)	$\bar{1}$		1	
(32)	$\bar{1}$	1	$\bar{1}$	
(31^2)	1			

WEIGHT 5. PARTITION (32).

	(5)	(32)
(32)		1
(3)(2)	1	1

	(32)	(3)(2)
(5)	$\bar{1}$	1
(32)	1	

WEIGHT 5. PARTITION (41).

	(5)	(41)
(41)		1
(4)(1)	1	1

	(41)	(4)(1)
(5)	$\bar{1}$	1
(41)	1	

WEIGHT 5. PARTITION (5).

	(5)
(5)	1

WEIGHT 6. PARTITION (1^6).

	(6)	(51)	(42)	(3^2)	(41^2)	(321)	(31^3)	(2^3)	(2^21^2)	(21^4)	(1^6)
(1^6)											1
$(1^5)(1)$										1	6
$(1^4)(1^2)$									1	4	15
$(1^3)^2$								1	2	6	20
$(1^4)(1)^2$							1		2	9	30
$(1^3)(1^2)(1)$						1	3	3	8	22	60
$(1^3)(1)^3$					1	3	10	6	18	48	120
$(1^2)^3$				1		3	6	6	15	36	90
$(1^2)^2(1)^2$			1	2	2	8	18	15	34	78	180
$(1^2)(1)^4$		1	4	6	9	22	48	36	78	168	360
$(1)^6$	1	6	15	20	30	60	120	90	180	360	720

	(1^6)	$(1^5)(1)$	$(1^4)(1^2)$	$(1^3)^2$	$(1^4)(1)^2$	$(1^3)(1^2)(1)$	$(1^3)(1)^3$	$(1^2)^3$	$(1^2)^2(1)^2$	$(1^2)(1)^4$	$(1)^6$
(6)	$\bar{6}$	6	6	3	$\bar{6}$	$\overline{12}$	6	$\bar{2}$	9	$\bar{6}$	1
(51)	6	$\bar{1}$	$\bar{6}$	$\bar{3}$	1	7	$\bar{1}$	2	$\bar{4}$	1	
(42)	6	$\bar{6}$	2	$\bar{3}$	2	4	$\bar{2}$	$\bar{2}$	1		
(3^2)	3	$\bar{3}$	$\bar{3}$	3	3	$\bar{3}$		1			
(41^2)	$\bar{6}$	1	2	3	$\bar{1}$	$\bar{3}$	1				
(321)	$\overline{12}$	7	4	$\bar{3}$	$\bar{3}$	1					
(31^3)	6	$\bar{1}$	$\bar{2}$		1						
(2^3)	$\bar{2}$	2	$\bar{2}$	1							
(2^21^2)	9	$\bar{4}$	1								
(21^4)	$\bar{6}$	1									
(1^6)	1										

WEIGHT 6. PARTITION (21^4)

	(6)	(51)	(42)	$2(3^2)$	(41^2)	(321)	(31^3)	$3(2^3)$	$2(2^21^2)$	(21^4)
(21^4)										1
$(21^3)(1)$							1		1	4
$(21^2)(1^2)+(1^4)(2)$						1	4	1	2	7
$(21)(1^3)$						1	3		1	4
$(21^2)(1)^2$					1	3	6	2	5	12
$(21)(1^2)(1)+(1^3)(2)(1)$			1	1	3	6	13	2	6	16
$(21)(1)^3$		1	4	3	6	13	18	6	12	24
$(2)(1^2)^2$			1		2	2	6		2	6
$(2)(1^2)(1)^2$		1	2	1	5	6	12	2	5	12
$(2)(1)^4$	1	4	7	4	12	16	24	6	12	24

÷ 30.

	(21^4)	$(21^3)(1)$	$(21^2)(1^2)+(1^4)(2)$	$(21)(1^3)$	$(21^2)(1)^2$	$(21)(1^2)(1)+(1^3)(2)(1)$	$(21)(1)^3$	$(2)(1^2)^2$	$(2)(1^2)(1)^2$	$(2)(1)^4$
(6)	30	$\overline{30}$	$\overline{30}$	$\overline{30}$	30	60	$\overline{30}$	30	$\overline{90}$	30
(51)	$\overline{30}$		30	30		$\overline{30}$		$\overline{30}$	30	
(42)	$\overline{30}$	30	$\overline{10}$	30	$\overline{10}$	$\overline{20}$	10	30	$\overline{10}$	
$2(3^2)$	$\overline{30}$	30	30	$\overline{60}$	$\overline{30}$	30		$\overline{30}$		
(41^2)	30		$\overline{10}$	$\overline{30}$	5	10	$\overline{5}$		5	
(321)	60	$\overline{30}$	$\overline{20}$	30	10	$\overline{4}$	2		$\overline{2}$	
(31^3)	$\overline{30}$		10		$\overline{5}$	2	$\overline{1}$		1	
$3(2^3)$	30	$\overline{30}$.	30	$\overline{30}$						
$2(2^21^2)$	$\overline{90}$	30	$\overline{10}$		5	$\overline{2}$	1		$\overline{1}$	
(21^4)	30									

WEIGHT 6. PARTITION (2^21^2). $\div 15$.

	(6)	(51)	(42)	(3^2)	(41^2)	(321)	$3(2^3)$	(2^21^2)
(2^21^2)								1
$(2^21)(1)$						1	1	2
$(21^2)(2)+(2^2)(1^2)$				1	1	2		3
$(21)^2$			1	2	2	2	2	4
$(2^2)(1)^2$			1	2		2	1	2
$(21)(2)(1)$		1	2	2	2	3	2	4
$(2)^2(1^2)$		1		2	1	2		2
$(2)^2(1)^2$	1	2	3	4	2	4	2	4

	(2^21^2)	$(2^21)(1)$	$(21^2)(2)+(2^2)(1^2)$	$(21)^2$	$(2^2)(1)^2$	$(21)(2)(1)$	$(2)^2(1^2)$	$(2)^2(1)^2$
(6)	$\overline{10}$	10	10	5	$\overline{10}$	$\overline{20}$	$\overline{10}$	15
(51)	10	5	$\overline{10}$	$\overline{5}$	$\overline{5}$	5	10	
(42)	10	$\overline{10}$	2	$\overline{5}$	4	3	$\overline{3}$	
(3^2)	5	$\overline{5}$	$\overline{5}$	5	5	$\overline{5}$	5	
(41^2)	$\overline{10}$	$\overline{5}$	4	5	$\overline{7}$	1	$\overline{1}$	
(321)	$\overline{20}$	5	3	$\overline{5}$	1	2	$\overline{2}$	
$3(2^3)$	$\overline{10}$	10	$\overline{3}$	5	$\overline{1}$	$\overline{2}$	2	
(2^21^2)	15							

WEIGHT 6. PARTITION (2^3).

	(6)	(42)	(2^3)
(2^3)			1
$(2^2)(2)$		1	3
$(2)^3$	1	3	6

	(2^3)	$(2^2)(2)$	$(2)^3$
(6)	3	$\overline{3}$	1
(42)	$\overline{3}$	1	
(2^3)	1		

WEIGHT 6. PARTITION (31^3).

	(6)	(51)	$2(3^2)$	(42)	(41^2)	(321)	(31^3)
(31^3)							1
$(31^2)(1)$					1	1	3
$(3)(1^3)$					1		1
$(31)(1^2)$				1	2	1	3
$(31)(1)^2$		1	1	2	4	3	6
$(3)(1^2)(1)$		1		1	3	1	3
$(3)(1)^3$	1	3	1	3	6	3	6

	(31^3)	$(31^2)(1)$	$(3)(1^3)$	$(31)(1^2)$	$(31)(1)^2$	$(3)(1^2)(1)$	$(3)(1)^3$
(6)	$\overline{1}$	1	1	1	$\overline{1}$	$\overline{2}$	1
(51)	1		$\overline{1}$	$\overline{1}$		1	
$2(3^2)$	1	$\overline{1}$	2	$\overline{1}$	1	$\overline{1}$	
(42)	1	$\overline{1}$	$\overline{1}$	1			
(41^2)	$\overline{1}$		1				
(321)	$\overline{2}$	1	$\overline{1}$				
(31^3)	1						

WEIGHT 6. PARTITION (321). ÷ 2.

	(6)	(51)	(42)	$2(3^2)$	(321)
(321)					1
(32)(1)			1	1	1
(31)(2)		1		1	1
(3)(21)		1	1		1
(3)(2)(1)	1	1	1	1	1

	(321)	(32)(1)	(31)(2)	(3)(21)	(3)(2)(1)
(6)	1	$\bar{1}$	$\bar{1}$	$\bar{1}$	2
(51)	$\bar{1}$	$\bar{1}$	1	1	
(42)	$\bar{1}$	1	$\bar{1}$	1	
$2(3^2)$	$\bar{1}$	1	1	$\bar{1}$	
(321)	1				

WEIGHT 6. PARTITION (3^2).

	(6)	(3^2)
(3^2)		1
$(3)^2$	1	2

	(3^2)	$(3)^2$
(6)	$\bar{2}$	1
(3^2)	1	

WEIGHT 6. PARTITION (41^2).

	(6)	(51)	(42)	(41^2)
(41^2)				1
(41)(1)		1	1	2
$(4)(1^2)$		1		1
$(4)(1)^2$	1	2	1	2

	(41^2)	(41)(1)	$(4)(1^2)$	$(4)(1)^2$
(6)	1	$\bar{1}$	$\bar{1}$	1
(51)	$\bar{1}$		1	
(42)	$\bar{1}$	1	$\bar{1}$	
(41^2)	1			

WEIGHT 6. PARTITION (42).

	(6)	(42)
(42)		1
(4)(2)	1	1

	(42)	(4)(2)
(6)	$\bar{1}$	1
(42)	1	

WEIGHT 6. PARTITION (51).

	(6)	(51)
(51)		1
(5)(1)	1	1

	(51)	(5)(1)
(6)	1	1
(51)	1	

WEIGHT 6. PARTITION (6).

	(6)
(6)	1

In these tables I have slightly varied the order of the partitions as I found it convenient in different cases; the proper order of the partitions for a table of given partitions remains, I think, to be discovered.

The sums of the powers of the roots may be at once written down in terms of the table separations by means of an extension of Waring's formula which I have previously given (loc. cit.), and it is convenient to repeat it here. If

$$(\lambda^l \mu^m \dots)$$

be any partition of n, and

$$(J_1)^{j_1}(J_2)^{j_2}\dots$$

one of its separations, then

$$(-)^{l+m+\dots}\frac{(l+m+\dots-1)!}{l!\,m!\dots}S_n = \sum(-)^{j_1+j_2+\dots}\frac{(j_1+j_2+\dots-1)!}{j_1!\,j_2!\dots}(J_1)^{j_1}(J_2)^{j_2}\dots,$$

where it will be observed that any coefficient depends merely upon the multiplicity of the separation which it multiplies, and that the right-hand side is a function of the same assemblages of separations as are employed, by rule, in the tables. This very important result is a good example of the extent and scope of the new theory; the mere existence of so suggestive a formula, of which the simplicity vies with the generality, points with a clearness which cannot be mistaken, to the conclusion that up to the present the algebraic theory of symmetric functions has been regarded from a point of view so little elevated that its chief beauties remained obscured. It is convincing proof that much may be expected from a comprehensive survey from the new vantage point.

General Remarks upon Syzygies.

In general the expression of any symmetric function in terms of the table separations is unique, but where we are not restricted to these combinations this is not so. There are as many table separations as there are species partitions of those separations, and in general the whole number of separations exceeds the number of corresponding species partitions; this difference indicates the number of syzygies which exist between the separations. Thus, turning to the table of partition (21^4) we observe 10 species partitions and 12 separations pointing to $12-10=2$ syzygies between the separations; the actual forms of the syzygies are easily obtained, for consider the product

$$(21^2)(2),$$

we may *either* express (21^2) by means of separations of (1^4) *or* (2) by means of separations of (1^2); in either way we arrive at the expression of the product by means of separations of (21^4). Thus

$$\begin{aligned}(21^2)(2) &= \{(1^3)(1)-4(1^4)\}(2)\\ &= (21^2)\{(1)^2-2(1^2)\}\end{aligned}$$

leading to the syzygy

$$2(21^2)(1^2)-(21^2)(1)^2+(1^3)(2)(1)-4(2)(1^4)=0;$$

the other syzygy is obtained from the product

$$(21)(2)(1)$$

and is found to be

$$2(21)(1^2)(1)-3(1^3)(2)(1)-(21)(1)^3+(2)(1^2)(1)^2=0.$$

More conveniently we may regard the first of these syzygies as arising from the partition (42),
for taking the product $(4)(2)$,

we may simultaneously express (4) in terms of separations of (1^4) and (2) in terms of separations of (2), *or* simultaneously (4) in terms of separations of (21^2) and (2) in terms of separations of (1^2).

In a similar manner the second syzygy arises from the partition

$$(321),$$

where observe that the partitions

$$(42),$$
$$(321)$$

are the only ones containing two different parts besides unity. So, considering the separations of

$$(21^{n-2})$$

and

$$\lambda_1^{l_1}\lambda_2^{l_2}\ldots\ldots\lambda_s^{l_s}1^t),$$

any one of its species partitions, we see that corresponding thereto there must be

$$s-1$$

syzygies, which may be written down according to the method above explained. The number of species partitions is the whole number of partitions less one, the generating function being

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots\ldots}-\frac{1}{1-x}.$$

If we write down all the partitions of a number and find that p_2 partitions contain a part 2, p_3 partitions a part 3, and so on, the number

$$p_2+p_3+\ldots\ldots+p_n$$

indicates the number of separations of

$$(21^{n-2});$$

the generating function for this number is

$$\frac{x^2}{(1-x)^2(1-x^2)(1-x^3)\ldots\ldots},$$

and hence the generating function for the syzygies between the separations of

$$(21^{n-2})$$

is

$$\frac{x^2}{(1-x)^2(1-x^2)(1-x^3)\ldots\ldots}-\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots\ldots}+\frac{1}{1-x},$$

which is

$$\frac{-1+x+x^2+(1-x)(1-x^2)(1-x^3)\ldots\ldots}{(1-x)^2(1-x^2)(1-x^3)\ldots\ldots},$$

or

$$\frac{\sum_{j=2}^{j=\infty}(-)^j(1+x^j)x^{\frac{1}{2}(3j^2-j)}}{(1-x)^2(1-x^2)(1-x^3)\ldots\ldots}.$$

The syzygies between the separations of other partitions may be worked out in a similar manner, though the calculations soon become laborious.

All these syzygies are linear relations between the separations of partitions, each syzygy involving only the separations of a single partition.

Beyond these there are syzygies connecting separations of different partitions which are at once seen by comparing the different tables of a given weight.

As regards any one table which expresses monomial symmetric functions in terms of separations of a given partition, any one row exhibits either such monomials as reducible quâ the separations, or else a congruence between the separations of highest degree.

By a comparison of the expressions of the same monomial symmetric function in different tables, syzygies are obtained between separations of different partitions; this circumstance I hope to discuss at some future time when applying the present theory to that of the covariants of binary forms.

Analogue of Newton's Theorem of the Sums of Powers.

The general expression of s_n in terms of the table separations of any partition of n was obtained by a comparison with a known particular case; for the proper discussion of the extended theory of symmetric functions brought forward here, it is necessary to see how the formula arises in another manner. We start with Newton's theorem for the expression of the sums of the powers in terms of the elementary coefficient, which is usually written as a series of relations, viz:

$$\begin{aligned}
&s_1-(1)=0,\\
&s_2-(1)s_1+2(1^2)=0,\\
&s_3-(1)s_2+(1^2)s_1-3(1^3)=0,\\
&s_4-(1)s_3+(1^2)s_2-(1^3)s_1+4(1^4)=0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

enabling the successive calculations of $s_2, s_3, s_4, \ldots.$ These relations are all exhibited by the single identity

$$\frac{(1)x-2(1^2)x^2+3(1^3)x^3-\ldots.}{1-(1)x+(1^2)x^2-(1^3)x^3+\ldots.}=s_1x+s_2x^2+s_3x^3+\ldots.,$$

for, on clearing of fractions and equating coefficients of like powers of x, the set of relations is produced.

Here, it is very important to observe, we can immediately obtain Waring's summation formula for the sums of the powers by expanding the denominator of the left-hand side by the multinomial theorem. This fact seems to have hitherto escaped the notice of writers upon the subject. This formula thus represents the expression of s_n in terms of separations of (1^n); the object in view is the corresponding formula for any other partition of n.

We can at once solve the problem for the partitions

$$(21^{n-2}),\ (31^{n-3}),\ (41^{n-4}),\ \dots\dots\ (\lambda 1^{n-\lambda})\ \dots\dots,$$

for, omitting the first of the series of relations, we may write

$$\begin{aligned}
&s_2 - (2) = 0,\\
&s_3 - (1)\, s_2 + (21) = 0,\\
&s_4 - (1)\, s_3 + (1^2)\, s_2 - (21^2) = 0,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

a series which leads to the identity

$$\frac{(2)\,x^2 - (21)\,x^3 + (21^2)\,x^4 - \dots\dots}{1 - (1)\,x + (1^2)\,x^2 - (1^3)\,x^3 + \dots\dots} = s_2 x^2 + s_3 x^3 + s_4 x^4 + \dots\dots,$$

and now expanding the sinister by the multinomial theorem we necessarily exhibit s_n in terms of separations of (21^{n-2}) and arrive at the corresponding summation formula.

Omitting the first of the last set of relations, we may write

$$\begin{aligned}
&s_3 - (3) = 0,\\
&s_4 - (1)\, s_3 + (31) = 0,\\
&s_5 - (1)\, s_4 + (1^2)\, s_3 - 31^2 = 0,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

and thence

$$\frac{(3)\,x^3 - (31)\,x^4 + (31^2)\,x^5 - \dots\dots}{1 - (1)\,x + (1^2)\,x^2 - (1^3)\,x^3 + \dots\dots} = s_3 x^3 + s_4 x^4 + s_5 x^5 + \dots\dots,$$

and proceeding in a similar manner we get finally

$$\frac{(\lambda)\,x^\lambda - (\lambda 1)\,x^{\lambda+1} + (\lambda 1^2)\,x^{\lambda+2} - \dots\dots}{1 - (1)\,x + (1^2)\,x^2 - (1^3)\,x^3 + \dots\dots} = s_\lambda x^\lambda + s_{\lambda+1} x^{\lambda+1} + s_{\lambda+2} x^{\lambda+2} + \dots\dots,$$

a formula which enables us to exhibit s_n by means of separations of $(\lambda 1^{n-\lambda})$. Put now

$$1 - (1)\,x + (1^2)\,x^2 - (1^3)\,x^3 + \dots\dots = (1 - \alpha x)(1 - \beta x)(1 - \gamma x)\dots\dots$$

and

$$\begin{aligned}
&\sum\nolimits_\lambda' = \sum \frac{\alpha^\lambda x^\lambda}{1 - \alpha x} = s_\lambda x^\lambda + s_{\lambda+1} x^{\lambda+1} + s_{\lambda+2} x^{\lambda+2} + \dots\dots,\\
&\sum\nolimits_\lambda'' = \sum \frac{\alpha^\lambda x^\lambda}{(1 - \alpha x)^2} = s_\lambda x^\lambda + 2 s_{\lambda+1} x^{\lambda+1} + 3 s_{\lambda+2} x^{\lambda+2} + \dots\dots,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\sum\nolimits_\lambda^{(m)} = \sum \frac{\alpha^\lambda x^\lambda}{(1 - \alpha x)^m} = s_\lambda x^\lambda + \frac{1}{2!}\, m\,(m + 1)\, s_{\lambda+1} x^{\lambda+1}\\
&\qquad\qquad + \frac{1}{3!}\, m\,(m + 1)(m + 2)\, s_{\lambda+2} x^{\lambda+2} + \dots\dots,
\end{aligned}$$

we have

$$\sum\nolimits'_{\lambda}\sum\nolimits'_{\mu}-\sum\nolimits''_{\lambda+\mu}=\sum\frac{\alpha^{\lambda}x^{\lambda}}{1-\alpha x}\sum\frac{\alpha^{\mu}x^{\mu}}{1-\alpha x}-\sum\frac{\alpha^{\lambda+\mu}x^{\lambda+\mu}}{(1-\alpha x)^{2}}$$

$$=\frac{\sum\alpha^{\lambda}\beta^{\mu}x^{\lambda+\mu}(1-\gamma x)(1-\delta x)}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)\,\ldots\ldots}$$

$$=\frac{(\lambda\mu)\,x^{\lambda+\mu}-(\lambda\mu 1)\,x^{\lambda+\mu+1}+(\lambda\mu 1^{2})\,x^{\lambda+\mu+2}-\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-(1^{3})\,x^{3}+\ldots\ldots},$$

or, taking previous results into account, this identity may be written

$$\frac{(\lambda)\,x^{\lambda}-(\lambda 1)\,x^{\lambda+1}+(\lambda 1^{2})\,x^{\lambda+2}-\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-\ldots\ldots}\cdot\frac{(\mu)\,x^{\mu}-(\mu 1)\,x^{\mu+1}+(\mu 1^{2})\,x^{\mu+2}-\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-\ldots\ldots}$$

$$-\frac{(\lambda\mu)\,x^{\lambda+\mu}-(\lambda\mu 1)\,x^{\lambda+\mu+1}+(\lambda\mu 1^{2})\,x^{\lambda+\mu+2}-\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-\ldots\ldots}$$

$$=s_{\lambda+\mu}x^{\lambda+\mu}+2s_{\lambda+\mu+1}x^{\lambda+\mu+1}+3s_{\lambda+\mu+2}x^{\lambda+\mu+2}+\ldots\ldots,$$

a formula which leads to the expression of s_n in terms of separations of $(\lambda\mu 1^{n-\lambda-\mu})$. Proceeding, we find

$$\sum\nolimits'_{\lambda}\sum\nolimits'_{\mu}\sum\nolimits'_{\nu}-\sum\nolimits'_{\lambda}\sum\nolimits''_{\mu+\nu}-\sum\nolimits'_{\mu}\sum\nolimits''_{\lambda+\nu}-\sum\nolimits'_{\nu}\sum\nolimits''_{\lambda+\mu}+2\sum\nolimits'''_{\lambda+\mu+\nu}$$

$$=\frac{(\lambda\mu\nu)\,x^{\lambda+\mu+\nu}-(\lambda\mu\nu 1)\,x^{\lambda+\mu+\nu+1}+(\lambda\mu\nu 1^{2})\,x^{\lambda+\mu+\nu+2}-\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-(1^{3})\,x^{3}+\ldots\ldots},$$

which should be compared and contrasted with the ordinary symmetric function formula

$$s_{\lambda}s_{\mu}s_{\nu}-s_{\lambda}s_{\mu+\nu}-s_{\mu}s_{\lambda+\nu}-s_{\nu}s_{\lambda+\mu}+2s_{\lambda+\mu+\nu}=(\lambda\mu\nu).$$

We are now led to the formula

$$\frac{(\lambda)\,x^{\lambda}-(\lambda 1)\,x^{\lambda+1}+\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-\ldots\ldots}\cdot\frac{(\mu)\,x^{\mu}-(\mu 1)\,x^{\mu+1}+\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-\ldots\ldots}\cdot\frac{(\nu)\,x^{\nu}-(\nu 1)\,x^{\nu+1}+\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-\ldots\ldots}$$

$$-\frac{1}{2}\,\frac{(\lambda)\,x^{\lambda}-(\lambda 1)\,x^{\lambda+1}+\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-\ldots\ldots}\cdot\frac{(\mu\nu)\,x^{\mu+\nu}-(\mu\nu 1)\,x^{\mu+\nu+1}+\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-\ldots\ldots}$$

$$-\frac{1}{2}\,\frac{(\mu)\,x^{\mu}-(\mu 1)\,x^{\mu+1}+\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-\ldots\ldots}\cdot\frac{(\lambda\nu)\,x^{\lambda+\nu}-(\lambda\nu 1)\,x^{\lambda+\nu+1}+\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-\ldots\ldots}$$

$$-\frac{1}{2}\,\frac{(\nu)\,x^{\nu}-(\nu 1)\,x^{\nu+1}+\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-\ldots\ldots}\cdot\frac{(\lambda\mu)\,x^{\lambda+\mu}-(\lambda\mu 1)\,x^{\lambda+\mu+1}+\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-\ldots\ldots}$$

$$+\frac{1}{2}\,\frac{(\lambda\mu\nu)\,x^{\lambda+\mu+\nu}-(\lambda\mu\nu 1)\,x^{\lambda+\mu+\nu+1}+\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-\ldots\ldots}$$

$$=s_{\lambda+\mu+\nu}x^{\lambda+\mu+\nu}+3s_{\lambda+\mu+\nu+1}x^{\lambda+\mu+\nu+1}+6s_{\lambda+\mu+\nu+2}x^{\lambda+\mu+\nu+2}+\ldots\ldots,$$

enabling us to express s_n in terms of separations of

$$(\lambda\mu\nu 1^{n-\lambda-\mu-\nu}).$$

From the mode in which these results have been evolved, it is absolutely certain, without further demonstration, that the expression

$$\frac{(\lambda\mu\nu\,\ldots\,)\,x^{\lambda+\mu+\nu+\ldots} - (\lambda\mu\nu\,\ldots\,1)\,x^{\lambda+\mu+\nu+\ldots+1} + \ldots}{1-(1)\,x+(1^2)\,x^2-\ldots}$$

is given in terms of

$$\sum\nolimits_{\lambda}', \sum\nolimits_{\mu}', \sum\nolimits_{\nu}', \ldots \sum\nolimits_{\lambda+\mu}'', \sum\nolimits_{\lambda+\nu}'', \sum\nolimits_{\mu+\nu}'', \ldots \sum\nolimits_{\lambda+\mu+\nu}''', \ldots$$

by precisely the same law as

$$(\lambda\mu\nu\,\ldots)$$

is given in terms of

$$s_{\lambda},\ s_{\mu},\ s_{\nu},\ \ldots\ s_{\lambda+\mu},\ s_{\lambda+\nu},\ s_{\mu+\nu},\ \ldots\ s_{\lambda+\mu+\nu},\ \ldots$$

If any number of equalities exist between the numbers

$$\lambda,\ \mu,\ \nu,\ \ldots$$

the same modifications nre requisite in both systems of formulae; accordingly both systems are inverted in a similar manner and following the same law; thus the formula above written is obtained by a comparison with the formula

$$(\lambda)(\mu)(\nu) - \frac{1}{2}(\lambda)(\mu\nu) - \frac{1}{2}(\mu)(\lambda\nu) - \frac{1}{2}(\nu)(\lambda\mu) + \frac{1}{2}(\lambda\mu\nu) = s_{\lambda+\mu+\nu};$$

in this manner is established the identity which for any separable partition is the direct analogue of the elementary formula

$$\frac{(1)\,x - 2\,(1^2)\,x^2 + 3\,(1^3)\,x^3 - \ldots}{1-(1)\,x+(1^2)\,x^2-(1^3)\,x^3+\ldots} = s_1x + s_2x^2 + s_3x^3 + \ldots,$$

and this leads by multinomial expansions to a proof of the summation formula which expresses s_n in terms of separations of any partition of n.

Derivation of Symmetric Functions from the Sums of Powers.

In the ordinary theory we derive the symmetric functions $(\lambda\mu)$, $(\lambda\mu\nu)$, ... from the formulae

$$(\lambda\mu) = s_{\lambda}s_{\mu} - s_{(\lambda+\mu)} \equiv s\,(1^{\lambda})\,s\,(1^{\mu}) - s\,(1^{\lambda+\mu}),^{*}$$

$$\begin{aligned}(\lambda\mu\nu) &= s_{\lambda}s_{\mu}s_{\nu} - s_{\lambda}s_{\mu+\nu} - s_{\mu}s_{\nu+\lambda} - s_{\nu}s_{\lambda+\mu} + 2s_{\lambda+\mu+\nu},\\ &\equiv s\,(1^{\lambda})\,s\,(1^{\mu})\,s\,(1^{\nu}) - s\,(1^{\lambda})\,s\,(1^{\mu+\nu}) - s\,(1^{\mu})\,s\,(1^{\nu+\lambda}) - s\,(1^{\nu})\,s\,(1^{\lambda+\mu})\\ &\quad + 2s\,(1^{\lambda+\mu+\nu}),\end{aligned}$$

---

[*] N. B. $s\,(\lambda\mu\,\ldots)$ denotes the expression of $s_{\lambda+\mu+\ldots}$ by means of separations of $(\lambda\mu\,\ldots)$.

and no difficulty is presented, because the products

$$s(1^\lambda)\,s(1^\mu),$$
$$s(1^\lambda)\,s(1^\mu)\,s(1^\nu),\ s(1^\lambda)\,s(1^{\mu+\nu}),\ s(1^\mu)\,s(1^{\nu+\lambda}),\ s(1^\nu)\,s(1^{\lambda+\mu}),$$

are, by direct multiplication, obtained in terms of separations of $(1^{\lambda+\mu})$, $(1^{\lambda+\mu+\nu})$ respectively. Had we been concerned with a separate partition composed of dissimilar parts, we could not thus have proceeded by direct multiplication; we could indeed have obtained the products in terms of separations, but the process would not have been a unique one and we would not have obtained an expression involving the tabular assemblages of separations. It is this expression that is required. Take the separable partion

$$(\lambda 1^{n-\lambda});$$

supposing it to be possible to express the product

$$s_p s_{n-p}$$

in terms of its tabular separations, we have, in the first place, two alternatives (in general). We may express s_p in terms of separations of (1^p), and s_{n-p} in terms of separations of $(\lambda 1^{n-p-\lambda})$, or *vice versa;* but neither of the products

$$s(1^p)\,s(\lambda 1^{n-p-\lambda}),$$
$$s(\lambda 1^{p-\lambda})\,s(1^{n-p})$$

are in terms of tabular separations of $(\lambda 1^{n-\lambda})$; we have in fact to take a linear function of these products. The investigation is facilitated by proceeding at once to the general case.

Let us then consider the problem of expressing the product

$$s_\lambda^l s_\mu^m \ldots.$$

by means of tabular separations of the partition

$$(t_1^{\tau_1} t_2^{\tau_2} t_3^{\tau_3} \ldots.).$$

Suppose $(\lambda_1^{l_1}\mu_1^{m_1}\ldots.)^{L_1}(\lambda_2^{l_2}\mu_2^{m_2}\ldots.)^{L_2}\ldots.$
to be any separation of $(t_1^{\tau_1} t_2^{\tau_2} t_3^{\tau_3} \ldots.)$,
having the species partition

$$(\lambda^l \mu^m \ldots.);$$

the general summation formula gives us

$$s(\lambda_1^{l_1}\mu_1^{m_1}\ldots.) = \ldots. + (-)^{l_1+m_1+\cdots+1}\frac{l_1!\ m_1!\ldots.}{(l_1+m_1+\ldots.-1)!}(\lambda_1^{l_1}\mu_1^{m_1}\ldots.);$$

hence, observing that

$$L_1(l_1+m_1+\dots\dots)+L_2(l_2+m_2+\dots\dots)+\dots\dots=\tau_1+\tau_2+\tau_3+\dots\dots$$

we find

$$\{s(\lambda_1^{l_1}\mu_1^{m_1}\dots\dots)\}^{L_1}\{s(\lambda_2^{l_2}\mu_2^{m_2}\dots\dots)\}^{L_2}\dots\dots$$

$$=\dots\dots+(-)^{\tau_1+\tau_2+\dots\dots+L_1+L_2+\dots\dots}\left\{\frac{l_1!\ m_1!\dots\dots}{(l_1+m_1+\dots\dots-1)!}\right\}^{L_1}$$

$$\left\{\frac{l_2!\ m_2!\dots\dots}{(l_2+m_2+\dots\dots-1)!}\right\}^{L_2}\dots\dots(\lambda_1^{l_1}\mu_1^{m_1}\dots\dots)^{L_1}(\lambda_2^{l_2}\mu_2^{m_2}\dots\dots)^{L_2}\dots\dots$$

Taking the summation for every separation having the species partition

$$(\lambda^l\mu^m\dots\dots),$$

we may write

$$\sum\left\{\frac{(l_1+m_1+\dots\dots-1)!}{l_1!\ m_1!\dots\dots}\right\}^{L_1}\left\{\frac{(l_2+m_2+\dots\dots-1)!}{l_2!\ m_2!\dots\dots}\right\}^{L_2}\dots\dots\frac{(L_1+L_2+\dots\dots-1)!}{L_1!\ L_2!\dots\dots}$$

$$\{s(\lambda_1^{l_1}\mu_1^{m_1}\dots\dots)\}^{L_1}\{s(\lambda_2^{l_2}\mu_2^{m_2}\dots\dots)\}^{L_2}\dots\dots$$

$$=\dots\dots+\sum(-)^{\tau_1+\tau_2+\dots\dots+L_1+L_2+\dots\dots}\frac{(L_1+L_2+\dots\dots-1)!}{L_1!\ L_2!\dots\dots}$$

$$(\lambda_1^{l_1}\mu_1^{m_1}\dots\dots)^{L_1}(\lambda_2^{l_2}\mu_2^{m_2}\dots\dots)^{L_2}\dots\dots$$

Observe that the dexter of this identity agrees with the corresponding portion of the expression of $s(t_1^{\tau_1}t_2^{\tau_2}\dots\dots)$, for

$$s(t_1^{\tau_1}t_2^{\tau_2}\dots\dots)=\dots\dots$$

$$+\sum(-)^{\tau_1+\tau_2+\dots+L_1+L_2+\dots}\frac{(L_1+L_2+\dots-1)!}{L_1!\ L_2!\dots}(\lambda_1^{l_1}\mu_1^{m_1}\dots)^{L_1}(\lambda_2^{l_2}\mu_2^{m_2}\dots)^{L_2}\dots+\dots,$$

and that we cannot obtain this agreement unless we take the linear function of the s products which appears on the sinister side; but we know that the s product

$$s_\lambda^l s_\mu^m\dots\dots$$

is necessarily expressible by means of tabular separations, because each of the monomial forms, which arise on performing the multiplication, is so; hence the above linear function of the s products must be a function of tabular separations of the partition $(t_1^{\tau_1}t_2^{\tau_2}\dots\dots)$.

We thus arrive at the result:

$$s_\lambda^l s_\mu^m\dots\dots=\frac{\displaystyle\sum\left\{\frac{(l_1+m_1+\dots\dots-1)!}{l_1!\ m_1!\dots\dots}\right\}^{L_1}\left\{\frac{(l_2+m_2+\dots\dots-1)!}{l_2!\ m_2!\dots\dots}\right\}^{L_2}\dots\dots\ \frac{(L_1+L_2+\dots\dots-1)!}{L_1!\ L_2!\dots\dots}\{s(\lambda_1^{l_1}\mu_1^{m_1}\dots\dots)\}^{L_1}\{s(\lambda_2^{l_2}\mu_2^{m_2}\dots\dots)\}^{L_2}\dots\dots}{\displaystyle\sum\left\{\frac{(l_1+m_1+\dots\dots-1)!}{l_1!\ m_1!\dots\dots}\right\}^{L_1}\left\{\frac{(l_2+m_2+\dots\dots-1)!}{l_2!\ m_2!\dots\dots}\right\}^{L_2}\dots\dots\frac{(L_1+L_2+\dots\dots-1)!}{L_1!\ L_2!\dots\dots}}$$

where

(i) the separable partition is $(t_1^{\tau_1}t_2^{\tau_2})$,

(ii) $(\lambda_1^{l_1}\mu_1^{m_1}\ldots.)^{L_1}(\lambda_2^{l_2}\mu_2^{m_2}\ldots.)^{L_2}\ldots.$ is any separation having the species partition $(\lambda^l\mu^m\ldots.)$,

(iii) the summations are in regard to all such separations; the result being that the dexter side is a function of tabular separations of $(t_1^{\tau_1}t_2^{\tau_2}\ldots.)$.

I illustrate the process by the calculation of (42) in terms of separations of $(2^2 1^2)$. We have

$$(42) = s_4 s_2 - s_6,$$

and

$$s_4 s_2 = \frac{\frac{1}{2}\cdot\frac{1}{2}\, s(2^2)\, s(1^2) + s(21^2)\, s(2)}{\frac{1}{2}\cdot\frac{1}{2} + 1} = \frac{1}{5}\, s(2^2)\, s(1^2) + \frac{4}{5}\, s(21^2)\, s(2)$$

$$= \frac{1}{5}\{-2(2^2)+(2)^2\}\{-2(1^2)+(1)^2\} + \frac{4}{5}\{(21^2)-(21)(1)-(2)(1^2)+(2)(1)^2\}(2),$$

$$= \frac{4}{5}(21^2)(2) + \frac{4}{5}(2^2)(1^2) - \frac{2}{5}(2^2)(1)^2 - \frac{6}{5}(2)^2(1^2) + (2)^2(1)^2,$$

which, since the separations $(21^2)(2)$, $(2^2)(1^2)$ occur with the same coefficient $\frac{4}{5}$, is, as it should be, a function of tabular separations.

Hence, extracting the value of $s_6 = s(2^2 1^2)$ from the tables, we find

$$\begin{aligned} 15(42) = {} & 10(2^2 1^2) - 10(2^2 1)(1) + 2\{(21^2)(2) + (2^2)(1^2)\} - 5(21)^2 \\ & + 4(2^2)(1)^2 + 8(21)(2)(1) - 8(2)^2(1^2). \end{aligned}$$

In this general way we are enabled, as in the ordinary theory, to combine the expressions for the sums of the powers so as to obtain the expressions of other symmetric functions by means of separations of any selected partition.

It will be gathered that this process would be very laborious in the case of symmetric functions whose partitions contained many parts. This is also the case in the ordinary or unitary theory (where by "unitary" it is meant that separable partitions composed wholly of parts, unity, are alone considered); it will be seen hereafter that this process will, for such forms, be naturally rejected in favor of easier methods of calculation.

Properties of Coefficients and Groups of Separations.

The general expression for the sum of the powers in terms of separations of any partition has, in regard to the numerical coefficients, a certain interesting and important property.

If the separable partition be (1^n) or (λ^n), it is a known theorem that the sum of the coefficients is

$$(-)^{n+1}.$$

Now, whenever the separable partition contains *dissimilar* parts, the sum of all the coefficients is *invariably* zero. This will be established by proving another theorem of a much more refined character from which it is immediately deducible. The separations of a given partition may be grouped in a manner which is independent of their species partitions. For clearness I take as a particular case the separable partition

$$(\lambda^3\mu^2)$$

and write down any one of its separations, say

$$(\lambda^2)(\lambda\mu)(\mu).$$

This separation may be regarded as compounded of the two separations:

$$(\lambda^2)(\lambda) \text{ of } (\lambda^3)$$

and

$$(\mu)^2 \text{ of } (\mu^2);$$

and moreover we will find that three other separations possess the same property, viz:

$$(\lambda^2)(\lambda)(\mu)^2,$$
$$(\lambda^2\mu)(\lambda)(\mu),$$
$$(\lambda^2\mu)(\lambda\mu);$$

for on suppressing the μ's in each we are left with

$$(\lambda^2)(\lambda),$$

and on suppressing the λ's there remains

$$(\mu)^2.$$

I say that these four separations

$$(\lambda^2)(\lambda\mu)(\mu),$$
$$(\lambda^2)(\lambda)(\mu)^2,$$
$$(\lambda^2\mu)(\lambda)(\mu),$$
$$(\lambda^2\mu)(\lambda\mu),$$

form a group, each member of which is compounded of the two separations

$$(\lambda^2)(\lambda),\ (\mu)^2.$$

I will call it the group $\{(\lambda^2)(\lambda),\ (\mu)^2\}$.

Consider now the separations of

$$(\lambda^3\mu^2)$$

as divided into six groups, viz:

$$\begin{array}{ll}
\text{The group} & \{(\lambda)^3 \quad\;\;, (\mu)^2\}, \\
\quad\text{"} & \{(\lambda)^3 \quad\;\;, (\mu^2)\}, \\
\quad\text{"} & \{(\lambda^2)(\lambda), (\mu)^2\}, \\
\quad\text{"} & \{(\lambda^2)(\lambda), (\mu^2)\}, \\
\quad\text{"} & \{(\lambda^3) \quad\;\;, (\mu)^2\}, \\
\quad\text{"} & \{(\lambda^3) \quad\;\;, (\mu^2)\},
\end{array}$$

where, be it observed, there is a group corresponding to every compound of a separation of (λ^3) with a separation of (μ^2).

So in general, if the separable partition be

$$(\lambda^l \mu^m \nu^n \ldots .),$$

we may take any separations of (λ^l), (μ^m), (ν^n) .... and form the group which is compounded of these separations. If (λ^l), (μ^m), (ν^n) .... possess L, M, N.... separations* respectively, there will thus be

$$LMN \ldots . \text{ groups.}$$

It will be observed that the grouping depends merely upon the multiplicities l, m, n, ... and not at all upon the parts λ, μ, ν, ...

I now enunciate the theorem:

Theorem: "In the expression of s_n in terms of separations of any partition of n, which contains dissimilar parts, the sum of the coefficients of the separations in each group is zero." I subjoin an example:

$$\begin{aligned}
\frac{3}{2}\, s\,(\lambda^2\mu^2) = {} & \frac{3}{2}\,(\lambda)^2(\mu)^2 - (\lambda)^2(\mu^2) + (\lambda^2)(\mu)^2 + (\lambda^2)(\mu^2) \\
& - 2\,(\lambda\mu)(\lambda)(\mu) + (\lambda\mu^2)(\lambda) + (\lambda^2\mu)(\mu) \; - (\lambda^2\mu^2) \\
& + \frac{1}{2}\,(\lambda\mu)^2 \\
\hline
& \pm 2 \qquad\qquad\quad \pm 1 \qquad\quad\; \pm 1 \qquad\quad\; \pm 1 \\
\hline
\end{aligned}$$

wherein the first, second, third, fourth columns of separations constitute the groups

$$\{(\lambda)^2, (\mu)^2\},\ \{(\lambda)^2, (\mu^2)\},\ \{(\lambda^2), (\mu)^2\},\ \{(\lambda^2), (\mu^2)\}$$

respectively, and the sum of the coefficients of the separations in each group is seen to be zero.

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* Or, what is the same thing, if the numbers l, m, n .... possess L, M, N.... partitions respectively.

To establish this theorem, consider in the first place the identity

$$\frac{(\lambda)x^{\lambda}-(\lambda 1)x^{\lambda+1}+(\lambda 1^2)x^{\lambda+2}-\ldots\ldots}{1-(1)x+(1^2)x^2-(1^3)x^3+\ldots\ldots}=s_{\lambda}x^{\lambda}+s_{\lambda+1}x^{\lambda+1}+s_{\lambda+2}x^{\lambda+2}+\ldots\ldots;$$

write the left-hand side in the form

$$\frac{(\lambda)y_{\lambda}-(\lambda 1)y_{\lambda}x_1+(\lambda 1^2)y_{\lambda}x_2-(\lambda 1^3)y_{\lambda}x_3+\ldots\ldots}{1-(1)x_1+(1^2)x_2-(1^3)x_3+\ldots\ldots},$$

wherein y_{λ}, x_t are symbolical representations of

x^{λ} and x^t respectively.

If this expression be developed by the multinomial theorem, we obtain a product

$$y_{\lambda}x_1^{l_1}x_2^{l_2}x_3^{l_3}\ldots\ldots$$

multiplied by a certain symmetric function. This function consists of a number of symmetric functions, each of which is a separation of

$$(\lambda 1^{l_1+2l_2+3l_3+\ldots\ldots})$$

and each of which is a member of the group of separations

$$\{(\lambda),\ (1)^{l_1}(1^2)^{l_2}(1^3)^{l_3}\ldots\ldots\}.$$

The complete symmetric function which multiplies

$$y_{\lambda}x_1^{l_1}x_2^{l_2}x_3^{l_3}\ldots\ldots$$

constitutes those terms in the expression of

$$s_{\lambda+l_1+2l_2+3l_3+\ldots\ldots}$$

which belong to the group of separations

$$\{(\lambda),\ (1)^{l_1}(1^2)^{l_2}(1^3)^{l_3}\ldots\ldots\}.$$

Putting
$$(\lambda)=(\lambda 1)=(\lambda 1^2)=\ldots\ldots=1,$$
$$(1)=(1^2)=\ldots\ldots=1,$$

the left-hand side of the identity reduces to

$$y_{\lambda};$$

hence, in the development by the multinomial theorem, all the other terms must vanish; but under these circumstances the aggregate of those separations which occur in any group of separations of a partition containing dissimilar parts, becomes the algebraic sum of the numerical coefficients which occur in such group; it follows that if s_n be expressed by means of separations of

$$(\lambda 1^{n-\lambda}) \qquad n>\lambda$$

the algebraic sum of the coefficients in each group must be zero.

Next consider the two identities

$$\frac{(\lambda)x^{\lambda}-(\lambda 1)x^{\lambda+1}+\ldots\ldots}{1-(1)x+(1^2)x^2-\ldots\ldots}\cdot\frac{(\mu)x^{\mu}-(\mu 1)x^{\mu+1}+\ldots\ldots}{1-(1)x+(1^2)x^2-\ldots\ldots}-\frac{(\lambda\mu)x^{\lambda+\mu}-(\lambda\mu 1)x^{\lambda+\mu+1}+\ldots\ldots}{1-(1)x+(1^2)x^2-\ldots\ldots}$$
$$=s_{\lambda+\mu}x^{\lambda+\mu}+2s_{\lambda+\mu+1}x^{\lambda+\mu+1}+3s_{\lambda+\mu+2}x^{\lambda+\mu+2}+\ldots\ldots,$$

$$\left\{\frac{(\lambda)x^{\lambda}-(\lambda 1)x^{\lambda+1}+\ldots\ldots}{1-(1)x+(1^2)x^2-\ldots\ldots}\right\}^2-2\,\frac{(\lambda^2)x^{2\lambda}-(\lambda^2 1)x^{2\lambda+1}+\ldots\ldots}{1-(1)x+(1^2)x^2-\ldots\ldots}$$
$$=s_{2\lambda}x^{2\lambda}+2s_{2\lambda+1}x^{2\lambda+1}+3s_{2\lambda+2}x^{2\lambda+2}+\ldots\ldots$$

These may, as regards their left-hand sides, be written symbolically

$$\frac{(\lambda)y_{\lambda}-(\lambda 1)y_{\lambda}x_1+(\lambda 1^2)y_{\lambda}x_2-\ldots\ldots}{1-(1)x_1+(1^2)x_2-(1^3)x_3+\ldots\ldots}\cdot\frac{(\mu)z_{\mu}-(\mu 1)z_{\mu}x_1+(\mu 1^2)z_{\mu}x_2-\ldots\ldots}{1-(1)x+(1^2)x_2-(1^3)x_3+\ldots\ldots}$$
$$-\frac{(\lambda\mu)y_{\lambda}z_{\mu}-(\lambda\mu 1)y_{\lambda}z_{\mu}x_1+\ldots\ldots}{1-(1)x+(1^2)x_2-\ldots\ldots}\left\{\frac{(\lambda)y_{\lambda}-(\lambda 1)y_{\lambda}x_1+(\lambda 1^2)y_{\lambda}x_2+\ldots\ldots}{1-(1)x_1+(1^2)x_2-(1^3)x_3+\ldots\ldots}\right\}^2$$
$$-2\,\frac{(\lambda^2)y_{2\lambda}-(\lambda^2 1)y_{2\lambda}x_1+(\lambda^2 1^2)y_{2\lambda}x_2-\ldots\ldots}{1-(1)x_1+(1^2)x_2-\ldots\ldots},$$

and herein putting all the symmetric functions equal to unity, we obtain respectively zero,

and $y_{\lambda}^2-2y_{2\lambda}$;

hence, when s_n is expressed by means of separations of

$$(\lambda\mu 1^{n-\lambda-\mu}),$$

the algebraic sum of the coefficients in each group of separations is zero, and also when s_n is expressed by means of separations of

$$(\lambda^2 1^{n-2\lambda}) \qquad n>2\lambda$$

the algebraic sum of the coefficients in each group is zero also.

This method of proof is perfectly general and leads to the important conclusion that whenever s_n is expressed by means of separations of a partition, which does not consist merely of repetitions of a single part, the algebraic sum of the coefficients in each group is zero.

In the proof it has been assumed that the algebraic sum of the coefficients on the dexter of such relation as

$$s(\lambda\mu\nu)=(\lambda)(\mu)(\nu)-\tfrac{1}{2}(\lambda\mu)(\nu)-\tfrac{1}{2}(\lambda\nu)(\mu)-\tfrac{1}{2}(\mu\nu)(\lambda)+\tfrac{1}{2}(\lambda\mu\nu),$$

is zero. It is easily seen that this is so, for writing down the relations

$$\begin{aligned}(\lambda) &= s(\lambda),\\ (\lambda\mu) &= s(\lambda)\,s(\mu) - s(\lambda\mu),\\ \text{etc.} &= \text{etc.}\end{aligned}$$

if we put $(\lambda) = (\lambda\mu) = \ldots . = 1$, we have

$$s(\lambda) = s(\mu) = 1,$$

and hence from the second relation

$$s(\lambda\mu) = 0.$$

Also the third relation being

$$(\lambda\mu\nu) = s(\lambda)\,s(\mu)\,s(\nu) - s(\lambda\mu)\,s(\nu) - s(\lambda\nu)\,s(\mu) - s(\mu\nu)\,s(\lambda) + 2s(\lambda\mu\nu),$$

putting

$$\begin{aligned}&(\lambda\mu\nu) = 1,\ s(\lambda) = s(\mu) = s(\nu) = 1,\\ &s(\lambda\mu) = s(\lambda\nu) = s(\mu\nu) = 0,\end{aligned}$$

we find $s(\lambda\mu\nu) = 0$, and so on in general, the expression for s_n becomes zero when, the separable partition containing dissimilar parts, the separates of the separations are each put equal to unity.

It is noteworthy that this property of the coefficients is not a generalization or extension of any known property, but is one which, by its nature, only comes into existence with the new theory; there is in fact no corresponding property in the ordinary theory, for in that theory the "group" does not exist *as a group*, but merely as an isolated term without interest. The theorem is of great importance as a verification of the coefficients of the separations, as well as for ascertaining that no separation has been accidentally omitted. It constitutes an example of departure from absolute generality.

This theory of the group may be easily extended to other symmetric functions.

Consider for a moment the expression of (3^2) by means of separations of (21^4). We have

$$2(3^2) = s_3^2 - s_6 = s(21)\,s(1^3) - s(21^4),$$

wherein we know that in both $s(21)$ and $s(21^4)$ the algebraic sum of the coefficients is zero; hence also in (3^2) the algebraic sum of all the coefficients must vanish; if, on the dexter side, there had been such a product as

$$s(1^4)\,s(2),$$

this could not have been the case, because the algebraic sum of the coefficients is not zero either in $s(1^4)$ or in $s(2)$. This s product does not occur simply because the partition (3^2) possesses no separation of species partition (42).

This reasoning is manifestly general, and we are thus led to the important theorem:

Theorem: "In the expression of symmetric function

$$(p_1^{\pi_1}p_2^{\pi_2}\dots\dots)$$

by means of separations of

$$(t_1^{\tau_1}t_2^{\tau_2}\dots\dots),$$

the algebraic sum of all the coefficients will be zero if the partition

$$(p_1^{\pi_1}p_2^{\pi_2}\dots\dots)$$

possesses no separations of species partition

$$(\tau_1 t_1,\ \tau_2 t_2)."$$

The theorem may be verified from the tables in the cases of

Separable partition.		Symmetric functions.
(21^4)	for	(51), (3^2),
(2^21^2)	for	(51), (3^2),
(31^3)	for	(51), (42), (41^2),
(321)	for	(51), (42), (3^2),
(41^2)	for	(51).

The group coefficient theory obtains also under the same circumstances; to establish this it will suffice to show that if we take any two sums of powers, say

$$s(\lambda^5\mu^4) \text{ and } s(\lambda^3\mu^2),$$

multiply out their product and arrange the resulting separations in groups, the algebraic sum of the coefficients in each group is zero. In $s(\lambda^5\mu^4)$ there is a group $\{(\lambda)^5, (\mu^2)(\mu)^2\}$, and in $s(\lambda^3\mu^2)$ a group $\{(\lambda^2)(\lambda), (\mu^2)\}$, and if we multiply the separations in these two groups together, we will, quâ the separable partitions $(\lambda^8\mu^6)$, obtain separations belonging to the group

$$\{(\lambda)^6(\lambda^2),\ (\mu^2)^2(\mu)^2\}.$$

In this batch of separations the algebraic sum of the coefficients is necessarily zero; we obtain another batch of separations comprised in the *same* group by multiplication of the groups

$$\{(\lambda)^3(\lambda^2),\ (\mu^2)^2\} \text{ and } \{(\lambda)^3,\ (\mu)^2\},$$

and again for this batch of separations the algebraic sum of the coefficients is zero. Observe that by thus multiplying certain *complete* groups of the one expression by certain *complete* groups of the other, we must obtain all the separations of $(\lambda^8\mu^6)$ belonging to the group

$$\{(\lambda)^6(\lambda^2),\ (\mu^2)^2(\mu)^2\}$$

which occur in the product $s(\lambda^5\mu^4)\,s(\lambda^3\mu^2)$. Hence, if in such product there occurs one factor at least of which the separable partition does not consist merely of repetitions of a single part, the algebraic sum of the coefficients in each group of the product will be zero. We are thus enabled to enunciate as follows:

Theorem: "In the expression of symmetric function

$$(p_1^{\pi_1}p_2^{\pi_2}\ldots.),$$

by means of separations of

$$(t_1^{\tau_1}t_2^{\tau_2}\ldots.),$$

the algebraic sum of the coefficients *in each group* will be zero, if the partition

$$(p_1^{\pi_1}p_2^{\pi_2}\ldots.)$$

possesses no separations of species partition

$$(\tau_1t_1,\ \tau_2t_2\ldots.)."$$

Concluding Remarks.

So far I have merely stood upon the threshold of the theory; the further development must be reserved for some future occasion; systems of partial differential equations exist analogous to those employed with such marked success by Brioschi, Hammond and others; and until this theory has been properly worked out, it does not seem advisable to enter upon the discussion of the calculation of the various tables.

The theory of those separable partitions which contain no part equal to unity is, from another point of view, a theory of the covariants of binary forms; in particular, in such cases the general theorem of algebraic reciprocity becomes a remarkable theorem of binary forms which promises to be chiefly of use in the discussion of perpetuants.

The "theorem of symmetric function expressibility" becomes a theorem, of a fundamental character, of covariant expressibility.

It is to be hoped that some of these facts may help to forward the algebraical (as distinct from the symbolical) treatment of the theory of invariants; as yet, however, a purely algebraical demonstration of Gordan's great theorem concerning the finality of the ground covariants seems as far distant as ever. This theorem has been given by Gordan, in a much simplified form, in a tract which appears to be little known; the reference was given to me by Mr. A. R. Forsyth of Trinity College, Cambridge, and it may be useful to give it here: "Ueber das Formensystem binärer Formen." Teubner (Leipzig, 1875).

ROYAL MILITARY ACADEMY, Woolwich, May 9th, 1888.

On the Integrals in Series of Binomial Differential Equations.

BY WILLIAM WOOLSEY JOHNSON.

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1. The term "binomial equation" was applied by Boole to the linear equation whose first member consists of two groups of terms, the terms of each group being homogeneous. Employing the notation

$$x\frac{d}{dx}=\vartheta,\text{ whence } x^2\frac{d^2}{dx^2}=\vartheta(\vartheta-1),\text{ etc.,}$$

the binomial equation may be written in the form

$$f(\vartheta)\,y-x^s\phi(\vartheta)\,y=X, \tag{1}$$

where f and ϕ are rational integral functions.

If the equation is of the second order, one at least of these functions is of the second degree, and the other of a degree not higher. Supposing both to be of the second degree, the binomial equation of the second order may be written

$$(\vartheta-a)(\vartheta-b)\,y-qx^s(\vartheta-c)(\vartheta-d)=X. \tag{2}$$

2. In integration by series it has been usual to consider only the case in which the second member is zero, in which case the substitution

$$y=\sum_0^\infty A_r x^{m+rs}$$

gives (since $\vartheta x^t=tx^t$),

$$\sum_0^\infty A_r\{(m+rs-a)(m+rs-b)\,x^{m+rs}-q\,(m+rs-c)(m+rs-d)\,x^{m+(r+1)s}\}=0,$$

which is satisfied if $\qquad A_0\,(m-a)(m-b)=0 \qquad (3)$

and, when $r>0$,

$$(m+rs-a)(m+rs-b)\,A_r-q\,(m+\overline{r-1}\,s-c)(m+\overline{r-1}\,s-d)\,A_{r-1}=0. \tag{4}$$

Equation (3) gives two values of m, and equation (4) determines the ratio between consecutive coefficients. Thus we have in general two independent integrals of the form $\sum_0^\infty A_r x^{a+rs}$ and $\sum_0^\infty B_r x^{b+rs}$ respectively. These series are convergent when $x<1$, if s is positive, and convergent when $x>1$, if s is negative. In like manner, we can obtain two series of the form $\sum_0^\infty A_r x^{c-rs}$ and $\sum_0^\infty B_r x^{d-rs}$, and these series will be convergent when the former series are divergent.

But if one of the functions f and ϕ is of the first degree, it is necessary to take for $f(\vartheta)$ that which is of the second degree, and we obtain two integrals of the form $\Sigma A_r x^{m+rs}$ which are always convergent, since by equation (4) the ratio $A_r : A_{r-1}$ will in this case contain two factors in the denominator which increase without limit, and only one such factor in the numerator. We can also obtain one series in the form Σx^{m-rs}, but it will be divergent for all values of x. In like manner, if the function ϕ is a mere constant, we have in general two integrals in the form of series convergent for all values of x.

It is noticeable that the trinomial equation of the second order does not thus always admit of integrals in the form of convergent series in x. Thus the equation being

$$f(\vartheta)\,y + x\phi(\vartheta)\,y + x^2\chi(\vartheta)\,y = 0,$$

the function $\phi(\vartheta)$ may be the only one of the second degree, and in that case we should have only divergent series either in ascending or in descending powers of x.

3. In discussing the binomial equation of the second order we may, without loss of generality, take q and s in equation (2) each equal to unity. For if we put

$$z = qx^s,\text{ whence } \vartheta' = z\frac{d}{dz} = \frac{\vartheta}{s},$$

the equation becomes

$$\left(\vartheta' - \frac{a}{s}\right)\left(\vartheta' - \frac{b}{s}\right)y - z\left(\vartheta' - \frac{c}{s}\right)\left(\vartheta' - \frac{d}{s}\right) = Z.$$

Taking therefore

$$(\vartheta - a)(\vartheta - b)\,y - x(\vartheta - c)(\vartheta - d)\,y = 0 \tag{5}$$

as the standard form when the second member is zero, the relation between consecutive coefficients is, from equation (4),

$$A_r = \frac{(m+r-1-c)(m+r-1-d)}{(m+r-a)(m+r-b)} A_{r-1}, \tag{6}$$

and, writing the complete integral in the form $y = A_0 y_1 + B_0 y_2$, we have the two independent integrals

$$\left.\begin{array}{l} y_1 = x^a\left[1 + \dfrac{(a-c)(a-d)}{1(a-b+1)}x + \dfrac{(a-c)(a-c+1)(a-d)(a-d+1)}{1.2(a-b+1)(a-b+2)}x^2 + \dots.\right] \\ \text{and} \\ y_2 = x^b\left[1 + \dfrac{(b-c)(b-d)}{1(b-a+1)}x + \dfrac{(b-c)(b-c+1)(b-d)(b-d+1)}{1.2(b-a+1)(b-a+2)}x^2 + \dots.\right], \end{array}\right\} (7)$$

or, in the notation of the hypergeometric series,

$$y_1 = x^a F(\alpha, \beta, \gamma, x),$$
$$y_2 = x^b F(\alpha', \beta', \gamma', x),$$

where

$$\begin{array}{ll} \alpha = a - c, & \alpha' = b - c, \\ \beta = a - d, & \beta' = b - d, \\ \gamma = a - b + 1, & \gamma' = b - a + 1. \end{array}$$

4. If we further transform equation (5) by putting $y = x^a v$, we have, since $f(\vartheta)\,x^a v = x^a f(\vartheta + a)\,v$,

$$\vartheta(\vartheta + a - b)\,v - x(\vartheta + a - c)(\vartheta + a - d)\,v = 0,$$

or

$$\vartheta(\vartheta + \gamma - 1)\,v - x(\vartheta + \alpha)(\vartheta + \beta)\,v = 0,$$

which is identical with the differential equation of the hypergeometric series as derived by Gauss from the properties of the series itself (Werke, Band III, p. 207). The binomial equation (2) is therefore reduced to Gauss' equation by the transformations $z = qx^s$, $y = z^{\frac{a}{s}} v$.

If the function ϕ in equation (1) is of the first degree, say $\phi(\vartheta) = q(\vartheta - c)$, q and s are each reduced to unity by putting x in place of $\frac{qx^s}{s}$, and we have then only to supply the factor $\frac{\vartheta - d}{-d}$ where d is infinite, to make the above general solution applicable; this is plainly equivalent to omitting the d-factors in equations (7). We have in this case

$$y_1 = x^a F\left(\alpha, \beta, \gamma, \frac{x}{\beta}\right),$$

where $\beta = \infty$, and x now stands for $\frac{qx^s}{s}$. In like manner, when $\phi(\vartheta) = q$, we omit also the c-factors, and we have

$$y_1 = x^a F\left(\alpha, \beta, \gamma, \frac{x}{\alpha\beta}\right),$$

where $\alpha = \infty$, $\beta = \infty$, and x stands for $\frac{qx^s}{s^2}$.

5. If one of the differences $a - c$, $a - d$, $b - c$, $b - d$ is zero or a negative integer, y_1 or y_2 becomes a finite series.

If $a = b$ the two series become identical, and if a and b differ by an integer, one of the integrals becomes infinite, a zero factor occurring in a denominator. We may in this last case assume a to be the greater of the two roots of f, so that $\gamma > 1$ and y_2 is the integral which fails. I have given elsewhere* the second integral in the case $\gamma \geqq 1$, in the form

$$y_2 = y_1 \log x + (-1)^\gamma \frac{(\gamma - 2)!\,(\gamma - 1)!}{(\alpha + 1 - \gamma)\dots(\alpha - 1)(\beta + 1 - \gamma)\dots(\beta - 1)}\, t + y', \tag{8}$$

where t is the sum of the terms of y_2 in equation (7) which precede the first infinite term (when $\gamma = 1$, $t = 0$), and

$$y' = x^a \Big[\frac{\alpha\beta}{1.\gamma}\Big(\frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{1} - \frac{1}{\gamma}\Big) x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}$$
$$\times \Big(\frac{1}{\alpha} + \frac{1}{\alpha+1} + \frac{1}{\beta} + \frac{1}{\beta+1} - \frac{1}{1} - \frac{1}{2} - \frac{1}{\gamma} - \frac{1}{\gamma+1}\Big) x^2 + \dots\Big], \tag{9}$$

a series differing from y_1 only in that each coefficient is multiplied by a quantity which may be called its *adjunct*, consisting of the sum of the reciprocals of the factors in the numerator taken positively and of those in the denominator taken negatively. The cases were also considered in which $a - b$ is an integer and at the same time one of the differences $a - c$, $a - d$ or $b - c$, $b - d$ is zero or a negative integer.

In the present paper the same method will be applied to the analogous special cases which arise in connection with the particular integral of the equation

$$(\vartheta - a)(\vartheta - b)\,y - x\,(\vartheta - c)(\vartheta - d)\,y = kx^p, \tag{10}$$

and the results will in fact include those previously found concerning the integrals y_1 and y_2 when the second member is zero.

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* The Messenger of Mathematics, Vol. XVI, p. 85, July, 1887.

6. Writing the complete integral of equation (10) in the form

$$y = A_0 y_1 + B_0 y_2 + Y,$$

y_1 and y_2 are the series given in equations (7), and putting $Y = \sum\limits_0^\infty C_r x^{m+r}$, we have, instead of equation (3),

$$C_0 (m-a)(m-b)\, x^m = kx^p,$$

whence $m = p$, and

$$C_0 = \frac{k}{(p-a)(p-b)}.$$

The relation between consecutive coefficients is still given by equation (6) and the result is

$$Y = \frac{kx^p}{(p-a)(p-b)} \Big[1 + \frac{(p-c)(p-d)}{(p-a+1)(p-b+1)}\, x + \frac{(p-c)(p-c+1)(p-d)(p-d+1)}{(p-a+1)(p-a+2)(p-b+1)(p-b+2)}\, x^2 + \ldots . \Big]. \quad (11)$$

If in the differential equation (10) we put $k = 0$, the exponent p is arbitrary; the value of Y in general reduces to zero; but, if we suppose $p = a$ or $p = b$, we have the product of an indeterminate coefficient into y_1 or y_2 as given in equations (7).

7. Among the special cases which arise we notice first that in which one of the differences $p - c$ or $p - d$ is zero or a negative integer. Y is in this case a finite series.

In the next place suppose one of the differences $p - a$ or $p - b$ to be a positive integer, say $p - a = i$. Then the $(i + 1)^{\text{th}}$ term of y_1 contains the same power of x that occurs in the first term of Y. Let M be a constant such that the $(i + 1)^{\text{th}}$ term of My_1 is identical with the first term of Y, then all the following terms will be identical, and the particular integral $Y' = Y - My_1$ will be a finite expression containing i terms. This finite integral would of course have been obtained directly if we had employed a descending series, in which case we should have written the differential equation in the form

$$(\vartheta - c)(\vartheta - d)\, y - x^{-1}(\vartheta - a)(\vartheta - b)\, y = -kx^{p-1},$$

and the particular integral would be

$$Y' = \frac{-kx^{p-1}}{(p-c-1)(p-d-1)} \Big[1 + \frac{(p-a-1)(p-b-1)}{(p-c-2)(p-d-2)}\, x^{-1} + \ldots . . \Big].$$

8. If $p-a$ or $p-b$ is zero or a negative integer, Y in equation (11) is infinite; thus, if $p-a=-i$, the coefficient of x^{p+i} contains a zero factor in the denominator. To find a particular integral having a finite value, we may put

$$p=a-i+h,$$

in which h will ultimately be put equal to zero. Denoting the sum of the first i terms by T, the value of Y is

$$Y=T+k\frac{(p-c)\ldots\ldots(p-c+i-1)(p-d)\ldots\ldots(p-d+i-1)}{(p-a)\ldots\ldots(p-a+i)(p-b)\ldots\ldots(p-b+i)}x^{p+i}$$
$$\times\Big[1+\frac{(p-c+i)(p-d+i)}{(p-a+i+1)(p-b+i+1)}x+\ldots\ldots\Big],$$

or

$$Y=T+k\frac{(p-c)\ldots\ldots(a-c-1+h)(p-d)\ldots\ldots(a-d-1+h)}{(-i+h)\ldots\ldots(-1+h)\,h\,(p-b)\ldots\ldots(a-b+h)}x^{a+h}$$
$$\times\Big[1+\frac{(a-c+h)(a-d+h)}{(1+h)(a-b+1+h)}x+\ldots\ldots\Big]. \quad (12)$$

Denoting the coefficient of x^{a+h} by $\frac{M}{h}$, and putting

$$\psi(h)=x^a\Big[1+\frac{(a-c+h)(a-d+h)}{(1+h)(a-b+1+h)}x$$
$$+\frac{(a-c+h)(a-c+1+h)(a-d+h)(a-d+1+h)}{(1+h)(2+h)(a-b+1+h)(a-b+2+h)}x^2+\ldots\ldots\Big], \quad (13)$$

the value of Y becomes

$$Y=T+\frac{M}{h}(1+h\log x+\ldots\ldots)[y_1+h\psi'(0)+\ldots\ldots], \quad (14)$$

since, by the first of equations (7), $y_1=\psi(0)$. Therefore, putting Y' for the particular integral $Y-\frac{M}{h}y_1$, we have

$$Y'=T+My_1\log x+M\psi'(0)+\ldots\ldots,$$

the terms not written involving powers of h.

When we put $h=0$, T in this equation is the sum of the terms in equation (11) which precede the first infinite term, M is the coefficient of this term with the zero factor in the denominator omitted, and, denoting $\psi'(0)$ by y', we have for the integral when $p=a-i$,

$$Y'=T+My_1\log x+My'. \quad (15)$$

In order to expand y' in powers of x, let us write

$$\psi(h)=\sum_0^\infty H_r x^{a+r};$$

so that, by equation (13),

$$H_0 = 1 \text{ and } H_r = \frac{(a-c+h)\ldots\ldots(a-c+r-1+h)(a-d+h)\ldots\ldots(a-d+r-1+h)}{(1+h)\ldots\ldots(r+h)(a-b+1+h)(a-b+r+h)};$$

then

$$\psi'(h) = \sum_0^\infty \frac{dH_r}{dh} x^{a+r} = \sum_1^\infty H_r \frac{d\log H_r}{dh} x^{a+r},$$

hence

$$\begin{aligned} y' = \psi'(0) = x^a \Big[& \frac{(a-c)(a-d)}{1\,(a-b+1)}\left(\frac{1}{a-c} + \frac{1}{a-d} - \frac{1}{1} - \frac{1}{a-b+1}\right) x \\ & + \frac{(a-c)(a-c+1)(a-d)(a-d+1)}{1.2\,(a-b+1)(a-b+2)}\left(\frac{1}{a-c} + \frac{1}{a-c+1} + \frac{1}{a-d}\right. \\ & \left. + \frac{1}{a-d+1} - \frac{1}{1} - \frac{1}{2} - \frac{1}{a-b+1} - \frac{1}{a-b+2}\right) x^2 \ldots\ldots \Big], \end{aligned} \qquad (16)$$

which is identical with equation (9), being formed from the value of y_1, equations (7), by the law of the adjuncts mentioned in §5.

9. If we put $k = 0$ and $p = b = a - i$, equation (15) gives the value of y_2 when a and b differ by an integer. Writing it in the form

$$Y' = M\left(y_1 \log x + y' + \frac{T}{M}\right),$$

and recurring to the values of T and M, see equations (11) and (12), we find that M is now indeterminate, but

$$\frac{T}{M} = -(-1)^i \frac{i!\,(i-1)!}{(b-c)\ldots\ldots(b-c+i-1)(b-d)\ldots\ldots(b-d+i-1)} x^b \times\left[1 + \frac{(b-c)(b-d)}{1\,(b-a+1)} x + \ldots\ldots\right];$$

we may therefore put, when $b = a - i$,

$$y_2 = y_1 \log x + y' - (-1)^i \frac{i!\,(i-1)!}{(b-c)\ldots\ldots(a-c-1)(b-d)\ldots\ldots(a-d-1)} t, \qquad (17)$$

where t stands for the sum of the terms which precede the first infinite term in y_2 as written in equations (7). Equation (17) is equivalent to the equation (8) quoted from a preceding paper.

10. The integral in equation (15) may be written out in accordance with the following rule: Write the terms as in equation (11) until a coefficient with a zero factor in the denominator is reached. Express the remainder of the integral as the product of this coefficient and a series whose first coefficient is unity. Omit the zero factor in the coefficient, and multiply each term in the series by the sum of log x and the adjunct of its coefficient.

11. Let us next suppose that, while a zero factor occurs in the denominator of a coefficient, one also occurs in the numerator of a coefficient; for example, suppose that $p-c$ as well as $p-a$ is zero or a negative integer. If the zero factor in the numerator occurs in the same or in an earlier coefficient than that in the denominator, no infinite coefficient occurs. Denoting the sum of the terms preceding that in which the zero factor occurs in the numerator by Y', Y is the sum of Y' and an indeterminate multiple of y_1, hence Y' is a particular integral. But if the zero factor first appears in the denominator, the integral is given by equation (15), and y_1 is a finite series. It is, however, to be noticed that y' is still an infinite series, for the adjunct of each of the vanishing coefficients in y_1 contains the reciprocal of the zero factor, and therefore simply destroys the zero factor, so that the remaining terms of y' are the vanishing terms of y_1 with the zero factor omitted. If we had regarded $\psi(h)$ as standing only for those terms in the second member of equation (13) which correspond to existing terms in y_1, we should have written $\psi(h)+h\chi(h)$ for the first member, and the final factor in equation (14) would have been

$$y_1+h\psi'(0)+h\chi(0)+\dots.$$

Equation (13) would then be obtained by putting $y'=\psi'(0)+\chi(0)$ in which $\psi'(0)$ now represents the existing terms of y_1 with adjuncts, and $\chi(0)$ the vanishing terms with the zero factor omitted. These terms in My' are the corresponding terms as they occur in Y, equation (11), with both zero factors omitted. Thus the rule for writing the integral given in §10 is in this case to be thus supplemented: If a zero factor occurs subsequently in a numerator, omit it, and discontinue the use of the logarithm and adjuncts.

12. Next, let us suppose that each of the differences $p-a$ and $p-b$ is zero or a negative integer; in this case two zero factors will occur in the denominators in equation (11). Of the two numbers a and b, let a be that which is not less than the other, we may then put

$$b=a-i \text{ and } p=b-j,$$

where i and j are positive integers or zero, and the coefficient of x^{p+j} is the first which is infinite. As before, we change the value of p to $b-j+h$, and ultimately put $h=0$.

Denoting, as in §8, by T the sum of the preceding terms, and by $\frac{M}{h}$ the coefficient which is infinite when $h=0$, the value of Y becomes

$$Y=T+\frac{M}{h}x^{p+j}\left[1+\frac{(p-c+j)(p-d+j)}{(p-a+j+1)(p-b+j+1)}x+\ldots.\right],$$

or

$$Y=T+\frac{M}{h}x^{h}.x^{b}\left[1+\frac{(b-c+h)(b-d+h)}{(b-a+1+h)(1+h)}x+\ldots.\right.$$

$$\left.+\frac{(b-c+h)\ldots(a-c-1+h)(b-d+h)\ldots(a-d-1+h)}{(-i+1+h)\ldots h(1+h)\ldots(i+h)}x^{i}\left\{1+\frac{(a-c+h)(a-d+h)}{(1+h)(i+1+h)}+\ldots\right\}\right].$$

If we write this in the form

$$Y=T+\frac{M}{h}x^{h}\left[t+\frac{\eta}{h}\right], \tag{18}$$

t is a function of h which, when $h=0$, becomes identical with t in equation (17) and

$$\eta=N\psi(h), \tag{19}$$

where $\psi(h)$ is the function defined in equation (13). Employing suffixes to denote the values assumed when $h=0$, we have

$$\eta_0=N_0\psi(0)=N_0y_1,$$

and it is readily seen from equation (17) that the complementary function in this case is

$$Ay_1+B\left[y_1\log x+y'+\frac{t_0}{N_0}\right].$$

Using accents to indicate differentiation with reference to h, equation (18) gives

$$Y=T+\frac{M}{h}(1+h\log x+\ldots.)(t_0+ht_0'+\ldots.)$$

$$+\frac{M}{h^2}\left(1+h\log x+\frac{1}{2}h^2\log^2 x+\ldots.\right)\left(\eta_0+h\eta_0'+\frac{1}{2}h^2\eta_0''+\ldots.\right),$$

or

$$Y=T+\frac{M}{h^2}\eta_0+\frac{M}{h}(t_0+\eta_0\log x+\eta_0')$$

$$+M\left(t_0\log x+t_0'+\frac{1}{2}\eta_0\log^2 x+\eta_0'\log x+\frac{1}{2}\eta_0''\right)+\ldots.,$$

the terms not written involving positive powers of h. From equation (19),

$$\eta_0'=N_0\psi'(0)+\psi(0)\left.\frac{dN}{dh}\right]_0=N_0y'+y_1\left.\frac{dN}{dh}\right]_0;$$

hence the terms involving negative powers of h are

$$\frac{MN_0}{h^2}y_1+\frac{MN_0}{h}\left[\frac{t_0}{N_0}+y_1\log x+y'\right]+\frac{M}{h}\left.\frac{dN}{dh}\right]_0y_1.$$

These terms coalesce with the complementary function above; hence, rejecting them and then putting $h = 0$, we may (now omitting the suffixes) write for the particular integral

$$Y' = T + M\left(t \log x + t' + \frac{1}{2}\eta \log^2 x + \eta' \log x + \frac{1}{2}\eta''\right). \qquad (20)$$

13. The series denoted by t' and η' are derived from t and η in the same manner that y' in §8 is derived from y_1, namely, by multiplying each coefficient by its adjunct. To find η'', let

$$\eta = \sum_0^\infty H_r x^{a+r},$$

then each H is of the form

$$\frac{(\alpha + h)(\beta + h) \ldots .}{(\lambda + h)(\mu + h) \ldots .}.$$

Differentiating

$$\frac{d\eta}{dh} = \sum_0^\infty H_r \frac{d \log H_r}{dh} x^{a+r} = \sum_0^\infty H_r \left[\sum \frac{1}{\alpha + h} - \sum \frac{1}{\lambda + h}\right] x^{a+r},$$

and

$$\frac{d^2\eta}{dh^2} = \sum_0^\infty H_r \left[\left\{\sum \frac{1}{\alpha + h} - \sum \frac{1}{\lambda + h}\right\}^2 - \sum \frac{1}{(\alpha + h)^2} + \sum \frac{1}{(\lambda + h)^2}\right] x^{a+r};$$

whence, putting $h = 0$,

$$\eta' = \sum_0^\infty H_r \left[\sum \frac{1}{\alpha} - \sum \frac{1}{\lambda}\right] x^{a+r} = \sum_0^\infty H_r (\text{adj})\, x^{a+r},$$

and

$$\eta'' = \sum_0^\infty H_r \left[(\text{adj})^2 - \sum \frac{1}{\alpha^2} + \sum \frac{1}{\lambda^2}\right] x^{a+r}. \qquad (21)$$

Thus η'' in equation (20) is the same series as that represented by η except that each coefficient is multiplied by a factor which may be called its *secondary adjunct*, which consists of the square of the adjunct together with the sum of the squares of the reciprocals of the factors in the denominator taken positively and of those in the numerator taken negatively.

14. The initial terms of equation (20) show that the integral may be written out in accordance with the rule given in §10 until the second zero factor is reached, and the final terms give the additional rule: After a second zero factor in a denominator is reached, omit this factor and multiply each term by the

sum of $\frac{1}{2}\log^2 x$, the product of $\log x$ by the adjunct of the coefficient, and one half of the secondary adjunct.

15. Finally, let us suppose that, while two zero factors occur in the denominators, a zero factor also occurs in a numerator, as in §11. If the occurrence of this factor is subsequent to that of the first but not of the second of those in the denominator, the particular integral ceases to take the form (20). In this case y_1 will be a finite series and y_2 will be of the regular form (7), containing higher powers of x than any which occur in y_1. The particular integral will be of the form (15) except that the use of the logarithm and adjuncts is discontinued in the interval between the appearance of the zero factor in the numerator and the second of those in the denominator; in other words, in the case of those powers of x which do not occur in either term of the complementary function.

16. If, however, the zero factor in the numerator occurs subsequently to both of those in the denominator, the particular integral is given by equation (20), but a factor α in one of the coefficients H and in every subsequent coefficient vanishes. The secondary adjunct in equation (21) may be written

$$\left(\frac{1}{\alpha}+\sum\frac{1}{\beta}-\sum\frac{1}{\lambda}\right)^2-\frac{1}{\alpha^2}-\sum\frac{1}{\beta^2}+\sum\frac{1}{\lambda^2},$$

or
$$2\,\frac{1}{\alpha}\left(\sum\frac{1}{\beta}-\sum\frac{1}{\lambda}\right)+\sum\left(\frac{1}{\beta}-\sum\frac{1}{\lambda}\right)^2-\sum\frac{1}{\beta^2}+\sum\frac{1}{\lambda^2}.$$

Hence one half the secondary adjunct is the infinite quantity

$$\frac{1}{\alpha}\left(\sum\frac{1}{\beta}-\sum\frac{1}{\lambda}\right),$$

and, since the adjunct itself is the infinite quantity $\frac{1}{\alpha}$, the factor by which every vanishing coefficient in η is multiplied is of the form

$$\frac{1}{\alpha}\left[\log x+\sum\frac{1}{\beta}-\sum\frac{1}{\lambda}\right].$$

The effect, therefore, is to cancel the vanishing factor and to multiply the term of η by the sum of $\log x$ and the adjunct of its coefficient as it now stands. This result was to have been expected, since, regarding the zero factor as cancelling one of those which have already occurred in the denominator, the subsequent terms may be regarded as belonging to the series t. In like manner, if a second zero factor occurs in a numerator (all four of the differences $p-a$, $p-b$, $p-c$ and $p-d$ being zero or negative integers), the terms return to the condition of the series T in equation (20), and are written as in equation (11), the four zero factors cancelling one another.

17. The complete rule for writing out the particular integral may be expressed as follows:

Write out the terms as in equation (11); let the series terminate if a zero coefficient occurs; in the infinite coefficients (which belong to powers of x occurring in one term of the complementary function), omit the zero factors and multiply by the sum of the logarithm and adjunct; and in the doubly infinite coefficients (which belong to powers of x occurring in both terms of the complementary function, in one of them with logarithms and adjuncts) multiply by the sum of the half square of the logarithm, the product of logarithm and adjunct, and the half secondary adjunct as in §14.

It is to be noticed that in forming the adjuncts and secondary adjuncts in equation (20), we ignore the factors included in the coefficient M. To include these factors would not indeed be incorrect, but would merely be equivalent to adding to Y' a quantity included in the complementary function.

18. In what precedes, the binomial equation is taken to be of the second order. The results are obviously similar when the equation is of the first order. Thus, supposing it reduced as in §3 to the standard form

$$(\vartheta - a)\,y - x\,(\vartheta - c)\,y = X,$$

the complementary function is

$$y_1 = x^a\left[1 + \frac{a-c}{1}\,x + \frac{(a-c)(a-c+1)}{1.2}\,x^2 + \ldots.\right]$$

[in finite form $y_1 = x^a\,(1-x)^{c-a}$]; and if $X = kx^p$, the particular integral is

$$Y = \frac{kx^p}{p-a}\left[1 + \frac{p-c}{p-a+1}\,x + \frac{(p-c)(p-c+1)}{(p-a+1)(p-a+2)}\,x^2 + \ldots.\right].$$

But if $p-a$ is zero or a negative integer, an infinite coefficient occurs and we have the form

$$Y' = T + My_1\log x + My',$$

as in equation (15), so that the rule for writing out the result is precisely the same as in §10.

19. It may also be noticed that the values of y_1 and Y above are the independent integrals of a special case of the equation of the second order when the second member is zero; for they are equivalent to the results of putting

$d = b - 1$ in equations (7). The case is that in which the equation admits of an integrating factor of the form x^m. Thus multiplying

$$(\vartheta - a)(\vartheta - b)\,y - x\,(\vartheta - c)(\vartheta - b + 1)\,y = 0$$

by x^{-b-1}, we have

$$x^{-b-1}(\vartheta - b)(\vartheta - a)\,y - x^{-b}(\vartheta - b + 1)(\vartheta - c)\,y = 0$$

which, by the theorem (C), p. 96 of the preceding volume of this Journal, may be written $\quad D\,[x^{-b}(\vartheta - a)\,y - x^{-b+1}(\vartheta - c)\,y] = 0,$

whence, integrating and multiplying by x^b, we have

$$(\vartheta - a)\,y - x\,(\vartheta - c)\,y = kx^b,$$

which is the binomial equation of the first order, its complementary function giving one, and its particular integral (involving k, which is now a constant of integration) furnishing the other of the two independent integrals of the given equation.

In like manner, the three series in equations (7) and (11) are the three independent integrals of a special case of the binomial equation of the third order.

20. The particular integral of the general binomial equation of the third order, when the second member is a power of x, is obviously of a form similar to Y in equation (11); thus, if the equation is reduced to the standard form

$$(\vartheta - a)(\vartheta - b)(\vartheta - c)\,y - x\,(\vartheta - \alpha)(\vartheta - \beta)(\vartheta - \gamma)\,y = kx^p,$$

the particular integral is

$$Y = \frac{kx^p}{(p-a)(p-b)(p-c)}\left[1 + \frac{(p-\alpha)(p-\beta)(p-\gamma)}{(p-a+1)(p-b+1)(p-c+1)}\,x + \ldots.\right]. \qquad (22)$$

The same rules also apply with reference to those terms whose coefficients, as written in accordance with this equation, would be infinite or doubly infinite. In fact, the demonstrations already given are applicable to binomial equations of any order.

21. For a term which would be triply infinite, the same method gives a similar extension of the rule. Thus, in place of equation (18), we have

$$Y = T + \frac{M}{h}\,x^h\left[t + \frac{\tau}{h} + \frac{\eta}{h^2}\right],$$

in which T, t, τ and η denote groups of terms written as in equation (22) except that the zero factors are omitted, there having been in the groups t, τ and η, an excess of one, two and three zero factors respectively in the denominator of each coefficient. In the development of this equation

$$
\begin{aligned}
Y = T &+ \frac{M}{h}(1 + h\log x + \ldots .)(t + ht' + \ldots .) \\
&+ \frac{M}{h^2}\left(1 + h\log x + \frac{1}{2}h^2\log^2 x + \ldots .\right)\left(\tau + h\tau' + \frac{1}{2}h^2\tau'' + \ldots .\right) \\
&+ \frac{M}{h^3}\left(1 + h\log x + \frac{1}{2}h^2\log^2 x + \frac{1}{6}h^3\log^3 x + \ldots .\right) \\
&\quad \times\left(\eta + h\eta' + \frac{1}{2}h^2\eta'' + \frac{1}{6}h^3\eta''' + \ldots .\right),
\end{aligned}
$$

the terms involving negative powers of h will be found to coalesce with the complementary function, so that the particular integral corresponding to equation (20) is the coefficient of h^0 in the development, that is

$$
\begin{aligned}
Y' = T + M\Big(t\log x + t' &+ \frac{1}{2}\tau\log^2 x + \tau'\log x + \frac{1}{2}\tau'' \\
&+ \frac{1}{6}\eta\log^3 x + \frac{1}{2}\eta'\log^2 x + \frac{1}{2}\eta''\log x + \frac{1}{6}\eta'''\Big). \quad (23)
\end{aligned}
$$

This equation shows that the rules already given still apply to the terms which, as first written, are singly and doubly infinite, and for those which are triply infinite gives the rule: Multiply each coefficient by $\frac{1}{6}\log^3 x$, the product of $\frac{1}{2}\log^2 x$ by the adjunct, the product of $\frac{1}{2}\log x$ by the secondary adjunct and one sixth of a *tertiary adjunct.*

22. The tertiary adjunct of a coefficient in η is the factor by which the coefficient must be multiplied to produce the corresponding coefficient of η'''. Employing the notation of §13 for any coefficient, thus

$$H = \frac{\alpha\beta\gamma \ldots .}{\lambda\mu\nu \ldots .},$$

let

$$
\begin{aligned}
S_1 &= \sum\frac{1}{\alpha} - \sum\frac{1}{\lambda}, \\
S_2 &= \sum\frac{1}{\lambda^2} - \sum\frac{1}{\alpha^2}, \\
S_3 &= \sum\frac{1}{\alpha^3} - \sum\frac{1}{\lambda^3}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
S_n &= (-1)^{n-1}\left[\sum\frac{1}{\alpha^n} - \sum\frac{1}{\lambda^n}\right].
\end{aligned}
$$

Denoting by D the operation of taking the derivative with respect to a quantity h, added to each of the quantities $\alpha, \beta, \gamma, \ldots.$ and $\lambda, \mu, \nu, \ldots.$ and afterwards put equal to zero,

$$DS_1 = S_2,\ DS_2 = 2S_3,\ DS_3 = 3S_4,\ \ldots.\ DS_n = nS_{n+1}.$$

We have then

$$\begin{aligned}
&DH = HS_1,\\
&D^2H = H(S_1^2 + S_2).\\
&D^3H = H(S_1^3 + 3S_1S_2 + 2S_3),\\
&D^4H = H(S_1^4 + 6S_1^2S_2 + 8S_1S_3 + 6S_4 + 3S_2^2),\\
&D^5H = H(S_1^5 + 10S_1^3S_2 + 20S_1^2S_3 + 30S_1S_4 + 24S_5 + 15S_1S_2^2 + 20S_2S_3),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

for the corresponding coefficients in $\eta', \eta'', \eta''', \ldots.$; thus, for example, the tertiary adjunct is $S_1^3 + 3S_1S_2 + 2S_3$.

23. In general, if η represent a group of terms in which as first written there is an excess of n zero factors in the denominator of each coefficient, the terms corresponding to η in the integral, derived as in §21, will obviously be the coefficient of h^0 in the development of

$$\frac{M}{h^n}\, e^{h \log x} . e^{hD} \eta\,;$$

that is to say, it is

$$\frac{M}{n!}(\log x + D)^n \eta .$$

24. As an example of the process, let us take the equation

$$(x^3 - x^4)\frac{d^3y}{dx^3} + (9x^2 - 2x^3)\frac{d^2y}{dx^2} + (18x + 2x^2)\frac{dy}{dx} + 6y = kx^{-5},$$

which, written in the ϑ notation, is

$$(\vartheta + 1)(\vartheta + 2)(\vartheta + 3)\,y - x\,(\vartheta - 2)\,\vartheta\,(\vartheta + 1)\,y = kx^{-5}.$$

By equation (22) we have

$$\begin{aligned}
Y = \frac{kx^{-5}}{-4.-3.-2} &+ \text{pr}\,\frac{-7.-5.-4}{-3.-2.-1}\,x^{-4} + \text{pr}\,\frac{-6.-4.-3}{-2.-1.0}\,x^{-3}\\
&+ \text{pr}\,\frac{-5.-3.-2}{-1.0.1}\,x^{-2} + \text{pr}\,\frac{-4.-2.-1}{0.1.2}\,x^{-1}\\
&+ \text{pr}\,\frac{-3.-1.0}{1.2.3}\,x^{0} + \text{pr}\,\frac{-2.0.1}{2.3.4}\,x^{1} + \text{pr}\,\frac{-1.1.2}{3.4.5}\,x^{2}\\
&+ \text{pr}\,\frac{0.2.3}{4.5.6}\,x^{3} + \text{pr}\,\frac{1.3.4}{5.6.7}\,x^{4} + \text{pr}\,\frac{2.4.5}{6.7.8}\,x^{5} + \ldots.,
\end{aligned}$$

in which, for shortness, the character pr in each coefficient is used to denote the preceding coefficient. The coefficient of x^r is here triply infinite only for $r = -1$; it is doubly infinite for $r = -2$ and $r = 0$, and singly infinite for $r = -3$, $r = 1$ and $r = 2$; the remaining terms belong to the group T. It is convenient to calculate beforehand the necessary values of S_1, S_2 and S_3, the values of these quantities being zero for the first infinite coefficient, and each successive value being found by adding to the preceding one the part corresponding to the new factors introduced. The result is given in the following table:

	r	S_1	S_2	S_3
$\frac{-5.-3.-2}{1.-1}$	-2	$-\frac{31}{30}$	$\frac{1439}{900}$	$-\frac{4591}{27000}$
$\frac{-4.-2.-1}{2.1}$	-1	$-\frac{31}{30}-\frac{13}{4}=-\frac{257}{60}$	$\frac{1439}{900}-\frac{1}{16}=\frac{5531}{3600}$	$-\frac{4591}{27000}-\frac{145}{64}=-\frac{526108}{216000}$
$\frac{-3.-1}{3.2.1}$	0	$-\frac{257}{30}-\frac{19}{6}=-\frac{149}{20}$	$\frac{5531}{3600}+\frac{1}{4}=\frac{6431}{3600}$	
$\frac{-2.-1}{4.3.2}$	1	$-\frac{149}{20}-\frac{7}{12}=-\frac{241}{30}$		
$\frac{-1.1.2}{5.4.3}$	2	$-\frac{241}{30}-\frac{17}{60}=-\frac{499}{60}$		

We may now write out the value of Y in accordance with the rules established, thus

$$
\begin{aligned}
Y' = & -\frac{k}{24}x^{-5}-\frac{35k}{36}x^{-4}+35k\Big[x^{-3}\log x+30x^{-2}\Big\{\frac{1}{2}\log^2 x-\frac{31}{30}\log x \\
& +\frac{1}{2}\Big(\frac{31^2}{30^2}+\frac{1439}{900}\Big)\Big\}-120x^{-1}\Big\{\frac{1}{6}\log^3 x+\frac{1}{2}\log^2 x\Big(-\frac{257}{60}\Big) \\
& +\frac{1}{2}\log x\Big(\frac{257^2}{60^2}+\frac{5531}{3600}\Big)+\frac{1}{6}\Big(-\frac{257^3}{60^3}-3\,\frac{257}{60}\,\frac{5531}{3600}-\frac{526103}{216000}\Big)\Big\} \\
& -60\Big\{\frac{1}{2}\log^2 x-\frac{149}{20}\log x+\frac{1}{2}\Big(\frac{149^2}{20^2}+\frac{6431}{3600}\Big)\Big\}+5x\Big(\log x-\frac{241}{30}\Big) \\
& -\frac{1}{6}x^2\Big(\log x-\frac{499}{60}\Big)-\frac{1}{120}x^3\Big(1+\frac{1.3.4}{5.6.7}x+\frac{1.3.4.2.4.5}{5.6.7.6.7.8}x^2+\ldots\Big)\Big],
\end{aligned}
$$

or finally

$$
\begin{aligned}
Y' = -\frac{k}{24}x^{-5}-\frac{35k}{36}x^{-4}+35k\Big[& x^{-3}\log x+15x^{-2}\log^2 x-31x^{-2}\log x+40x^{-2} \\
& -20x^{-1}\log^3 x+257x^{-1}\log^2 x-1193x^{-1}\log x \\
& +\frac{21765097}{1296000}x^{-1}-30\log^2 x+447\log x-\frac{5156}{3} \\
& +5x\log x-\frac{241}{6}x-\frac{1}{6}x^2\log x+\frac{499}{360}x^2 \\
& -\frac{1}{120}x^3-\frac{1}{2100}x^4-\ldots\Big].
\end{aligned}
$$

If $Ay_1 + By_2 + Cy_3$ be the complementary function in this example, the values of y_1, y_2 and y_3 may be derived in like manner from the value of Y above, taking the coefficients of y_{-1}, y_{-2} and y_{-3} respectively as unity. We should thus find

$$y_3 = x^{-3} + 30x^{-2}\log x - 60x^{-1}\log^2 x + 390x^{-1}\log x - 630x^{-1} - 60\log x + 385 + 5x - \frac{1}{6}x^2,$$

$$y_2 = x^{-2} - 4x^{-1}\log x - 2,$$

and

$$y_1 = x^{-1}$$

(so that the terms containing x^{-1} in the values of Y' and y_3 may be rejected).

25. The following formulae for the verification of integrals consisting of terms of the form x^r, $x^r\log x$, $x^r\log^2 x$, etc., may be noticed in conclusion. From the definition of the symbol ϑ we have

$$\vartheta x^r V = x^r(\vartheta + r)V, \quad \vartheta^2 x^r V = \vartheta x^r(\vartheta + r)V = x^r(\vartheta + r)^2 V, \text{ etc.}$$

whence $f(\vartheta)$ being an algebraic function,

$$f(\vartheta)x^r V = x^r f(\vartheta + r)V.$$

But since

$$\begin{aligned} &\vartheta 1 = 0, \\ &\vartheta\log x = 1, \qquad \vartheta^2\log x = 0, \\ &\vartheta\log^2 x = 2\log x, \quad \vartheta^2\log^2 x = 2, \quad \vartheta^3\log^2 x = 0, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}$$

we have

$$\begin{aligned} f(\vartheta)1 &= f(0), \\ f(\vartheta)\log x &= f'(0) + f(0)\log x, \\ f(\vartheta)\log^2 x &= f''(0) + 2f'(0)\log x + f(0)\log^2 x, \\ f(\vartheta)\log^3 x &= f'''(0) + 3f''(0)\log x + 3f'(0)\log^2 x + f(0)\log^3 x, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}$$

Hence, putting in the formula above $V = \log^n x$, we have for $n = 0, 1, 2, \dots$

$$\begin{aligned} f(\vartheta)x^r &= x^r f(r), \\ f(\vartheta)x^r\log x &= x^r[f'(r) + f(r)\log x], \\ f(\vartheta)x^r\log^2 x &= x^r[f''(r) + 2f'(r)\log x + f(r)\log^2 x], \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}$$

and in general,

$$f(\vartheta)x^r\log^n x = \left[\frac{d}{d\vartheta} + \log x\right]^n f(\vartheta)x^r.$$

Now if the differential equation be

$$f(\vartheta)y - x\phi(\vartheta)y = kx^p,$$

the integral, which is of the form

$$Y = \Sigma A_r x^r + \log x \Sigma B_r x^r + \log^2 x \Sigma C_r x^r + \ldots.,$$

will be verified if the result of operating upon it with

$$f(\vartheta) - x\phi(\vartheta)$$

is kx^p. The result of operation is, by the formulae just demonstrated,

$$\begin{aligned}
&\Sigma A_r f(r) x^r \;\; - \Sigma A_r \phi(r) x^{r+1} \\
+ \;&\Sigma B_r f'(r) x^r - \Sigma B_r \phi'(r) x^{r+1} + [\Sigma B_r f(r) x^r - \Sigma B_r \phi(r) x^{r+1}] \log x \\
+ \;&\Sigma C_r f''(r) x^r - \Sigma C_r \phi''(r) x^{r+1} + 2[\Sigma C_r f'(r) x^r - \Sigma C_r \phi'(r) x^{r+1}] \log x \\
+ \;&[\Sigma C_r f(r) x^r \;\; - \Sigma C_r \phi(r) x^{r+1}] \log^2 x, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

in which the aggregate of terms independent of $\log x$ must reduce to kx^p, and the coefficient of each power of $\log x$ must separately vanish. The first of these conditions involving all the terms of Y will constitute a good verification; and it appears that the aggregate which should reduce to kx^p consists simply of the products of each term independent of $\log x$ by the value of $f(r) - x\phi(r)$ for that term, of each term in the coefficient of $\log x$ by the value of $f'(r) - x\phi'(r)$, and so on. The example in §24 was verified in this way.

The last of the conditions mentioned above shows that the coefficient of the highest power of $\log x$ in Y is in all cases a multiple of a term of the complementary function.

Sur certaines courbes qu'on peut adjoindre aux courbes planes pour l'étude de leurs propriétés infinitésimales.[*]

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1. Soit c une courbe plane quelconque. Dans le plan de cette courbe prenons arbitrairement deux points O et P (Fig. 1). M étant un point variable sur la courbe c, tirons OM et menons PH parallèlement à la normale en M à la courbe c. Le lieu du point H est une courbe que nous désignerons par la lettre n.

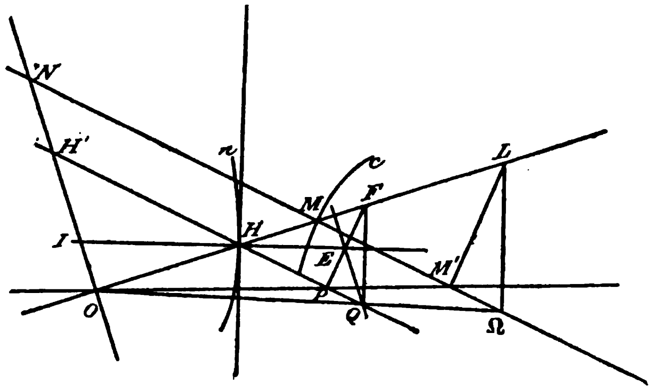

FIG. 1.

On voit immédiatement que la courbe n passe par les pieds des normales menées du point P à la courbe c, qu'elle a pour asymptotes les normales menées à la même courbe par le point O, et qu'elle passe par les points de rencontre du cercle décrit sur OP comme diamètre, avec les tangentes menées de O à la courbe c et avec les parallèles aux asymptotes de la courbe c menées par le point O.

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[*] Mémoire présenté à l'Académie des Sciences de Lisbonne.

Si la courbe n est connue *à priori*, elle détermine du même coup tous les éléments qui viennent d'être énumérés; elle permet en outre évidemment de mener la normale en un point donné sur la courbe c ou de déterminer les points de cette courbe où la normale a une direction donnée.

Voyons maintenant comment elle permet aussi de construire le centre de courbure répondant à tout point de la courbe c.

2. Soient Ω, le centre de courbure relatif au point M,

N et I, les points où la perpendiculaire élevée en O à OM coupe les normales en M et en H aux courbes c et n,

E, le point où la perpendiculaire élevée en P à PH coupe la normale HI.

Représentant par dc et dn les différentielles des arcs des courbes c et n aux points M et H, par $d\omega$ et da les différentielles des angles que OM et PH font avec OP, on a

$$dc = MN.d\omega = M\Omega.da,$$

$$dn = HI.d\omega = HE.da.$$

Donc

$$\frac{M\Omega}{MN} = \frac{HE}{HI}.$$

Par le point E menons à OM une perpendiculaire qui coupe HP au point Q, et appelons H' le point où HP coupe OI. Nous avons, par les triangles semblables HEQ et HIH',

$$\frac{HE}{HI} = \frac{HQ}{HH'}$$

par suite,

$$\frac{HQ}{HH'} = \frac{M\Omega}{MN},$$

ce qui prouve que le point Q se trouve sur la droite $O\Omega$. Cette remarque permet d'obtenir le point Ω, mais nous allons simplifier la construction. A cet effet, prolongeons PE jusqu'à sa rencontre en F avec OM, et tirons QF.

Dans le triangle EHQ, EP est la hauteur issue du sommet E, HF la hauteur issue du sommet H; il suit de là que QF est la hauteur issue du sommet Q; et, comme HE est la normale en H à la courbe n, on voit que QF est parallèle à la tangente en H à cette courbe.

Soit M' le point où la normale en M à la courbe c coupe la droite fixe OP. En M' élevons à MM' une perpendiculaire qui coupe OM en L. PF étant perpendiculaire à PH est parallèle à $M'L$. Donc les triangles OLM' et OFP sont semblables, et on a

$$\frac{OL}{OF} = \frac{OM'}{OP} = \frac{O\Omega}{OQ}.$$

De là résulte que $L\Omega$ est parallèle à QF, et, par suite, à la tangente en H à la courbe n.

La construction du centre de courbure Ω relatif au point M est donc, en définitive, la suivante :

La normale en M à la courbe c coupant la droite fixe OP au point M', on élève en M' à MM' une perpendiculaire qui coupe le rayon vecteur OM au point L ; par le point L on mène à la tangente en H à la courbe n une parallèle qui coupe MM' au centre de courbure Ω.

3. Si PH est tangente en H à la courbe n, $L\Omega$ est parallèle à MM'. Le centre de courbure Ω est donc rejeté à l'infini et le point M correspondant est un point d'inflexion. Donc,

Les points H de la courbe n correspondant aux points d'inflexion de la courbe c sont les points de contact des tangentes menées de P à la courbe n.

4. Les propriétés sus-énoncées de la courbe n montrent tout l'intérêt qui s'attache à sa considération pour l'étude de la courbe c. Dès lors une question se pose tout naturellement à l'esprit: *La courbe n étant une des courbes les plus simples, droite ou cercle, trouver toutes les courbes c correspondantes.*

Remarquons d'abord que l'équation différentielle des courbes c correspondant à une courbe n donnée se forme immédiatement. Prenons O pour origine, OP pour axe des x et soient x et y les coordonnées du point M, x_1 et y_1 celles du point H; on a d'abord

$$\frac{y}{x} = \frac{y_1}{x_1}, \tag{1}$$

et, en posant $OP = a$,

$$\frac{dy}{dx} = \frac{a - x_1}{y_1}. \tag{2}$$

Si donc l'équation de la courbe n est

$$f(x_1, y_1) = 0,$$

l'équation différentielle des courbes c correspondantes s'obtiendra par l'élimination de x_1 et y_1 entre les trois équations précédentes.

5. Soit donc d'abord

$$y_1 = mx_1 + n \tag{3}$$

l'équation de la courbe n. Des équations (1) et (2) on tire

$$x_1 = \frac{axdx}{xdx + ydy},$$

$$y_1 = \frac{aydx}{xdx + ydy}.$$

Portant ces valeurs dans (3), on a

$$aydx = maxdx + n(xdx + ydy)$$

ou

$$[(ma + n)x - ay]\, dx + nydy = 0,$$

équation homogène dans laquelle on séparera les variables par la substitution $y = ux$, ce qui donne

$$\frac{dx}{x} + \frac{udu}{u^2 - \frac{a}{n}u + \frac{ma}{n} + 1} = 0. \tag{4}$$

Il y a trois cas à considérer pour l'intégration de cette équation suivant la nature des racines du trinôme du second degré en u, qui figure en dénominateur.

1° Cas.—L'équation

$$u^2 - \frac{a}{n}u + \frac{ma}{n} + 1 = 0,$$

a ses deux racines réelles.

Soient p et q ces deux racines, p étant $> q$.

On a alors

$$\frac{u}{u^2 - \frac{a}{n}u + \frac{ma}{n} + 1} = \frac{p}{p-q}\cdot\frac{1}{u-p} - \frac{q}{p-q}\cdot\frac{1}{u-q},$$

et l'équation (4) devient

$$\frac{dx}{x} + \frac{p}{p-q}\cdot\frac{du}{u-p} - \frac{q}{p-q}\cdot\frac{du}{u-q} = 0,$$

d'où, en intégrant

$$x^{p-q}\frac{(u-p)^p}{(u-q)^q} = c,^*$$

ou

$$\frac{(y-px)^p}{(y-qx)^q} = c. \tag{5}$$

On obtient donc des courbes algébriques si p et q sont commensurables.

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* Ici, comme dans tout le reste du Mémoire, nous désignons par c une constante arbitraire; nous conserverons par suite cette lettre dans les transformations successives d'une même formule, bien que si on suppose attribuée une certaine valeur à cette constante dans une des formules, cette valeur puisse ne pas être la même pour les constantes représentées par la même lettre dans les formules subséquentes.

Quels que soient p et q, la courbe (5) coupe normalement l'axe Ox; la vérification de ce fait est immédiate.

Supposons que p et q soient entiers.

Si p et q sont de même signe, la courbe appartient au genre parabolique c'est-à-dire qu'elle a des branches infinies sans asymptotes. Elle est tangente à l'origine à la droite $y = qx$.

Si p et q sont des signes contraires, la courbe appartient au genre hyperbolique. Elle admet pour asymptotes les droites $y = px$ et $y = qx$. Des relations

$$p + q = \frac{\alpha}{n}, \quad pq = \frac{m\alpha}{n} + 1,$$

on tire

$$m = \frac{pq - 1}{p + q}, \quad n = \frac{\alpha}{p + q}.$$

Le coefficient m étant indépendant de α, si le point P se déplace sur l'axe Ox, la droite n pour la même courbe (5) se déplace parallèlement à elle-même. Or, d'après une des propriétés fondamentales des courbes n (No. 1), les points de rencontre de la droite n et de la courbe (5) sont les pieds des normales menées à cette courbe du point P. Ainsi donc:

Lorsque le point P décrit l'axe Ox, la droite qui joint les pieds des normales menées de ce point à la courbe (5) *se déplace parallèlement à elle-même.*

En particulier pour $p = 2$, $q = 1$, la courbe (5) devient la parabole

$$\frac{(y - 2x)^2}{y - x} = c.$$

On a donc ce théorème:

Si un point décrit une normale à une parabole, on peut de ce point mener constamment deux autres normales à la parabole. La droite qui joint les pieds de ces deux normales se déplace parallèlement à elle-même.

On peut aussi remarquer, en vertu d'une autre propriété fondamentale des courbes n, que la droite n passe par les points de rencontre du cercle décrit sur OP comme diamètre avec les tangentes menées de O à la courbe (5).

Lorsque cette courbe appartient au genre parabolique, ces tangentes se confondent avec la tangente en O à la courbe (5). Donc, dans ce cas:

La droite n est la tangente au cercle décrit sur OP comme diamètre menée par le point où ce cercle est coupé par la tangente en O à la courbe (5), *c'est-à-dire par la droite $y = qx$.*

En particulier pour la parabole répondant aux valeurs $p = 2$, $q = 1$, on a ce théorème :

La normale en un point d'une parabole coupe cette courbe en un second point O. D'un point P pris sur cette normale on peut mener à la parabole deux autres normales. Les pieds de ces normales sont sur la tangente au cercle décrit sur OP comme diamètre, menée par le point où ce cercle est coupé par la tangente en O à la parabole.

Dans le cas où la courbe (5) appartient au genre hyperbolique les tangentes issues de O sont les asymptotes $y = px$ et $y = qx$. On a alors ce théorème :

La droite n est la droite qui joint les points de rencontre du cercle décrit sur OP comme diamètre avec les asymptotes de la courbe (5).

Un cas particulier intéressant est celui où l'on a $q = -p$, qui se produit si la droite n est perpendiculaire à OP.

L'équation (5) devient alors

$$y^2 - p^2x^2 = c;$$

conique de centre O ayant ses axes dirigés suivant Ox et Oy. Ainsi :

Le lieu des points de rencontre des diamètres d'une conique et des parallèles aux normales correspondantes, menées par un point de l'un les axes, est une perpendiculaire à cet axe.

Si nous appliquons ici la construction du centre de courbure donnée au No. 2, nous obtenons précisément la construction qui a été indiqué pour l'ellipse par M. Mannheim* et qu'il a obtenue par une voie essentiellement différente de celle qui précède. Il est curieux de voir que cette dernière construction résulte simplement d'un cas très particulier d'une propriété extrêmement générale.

2ᵉ Cas.—L'équation

$$u^2 - \frac{a}{n}u + \frac{ma}{n} + 1 = 0$$

a ses racines égales.

Soit p cette racine double. On a alors

$$\frac{u}{u^2 - \frac{a}{n}u + \frac{ma}{n} + 1} = \frac{p}{(u-p)^2} + \frac{1}{u-p},$$

et l'équation (4) devient

$$\frac{dx}{x} + \frac{pdu}{(u-p)^2} + \frac{du}{u-p} = 0.$$

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* *Cours de géométrie descriptive de l'Ecole polytechnique*, 1ᵉ éd., p. 175 ; 2ᵉ éd., p. 173.

Son intégrale est

$$lx - \frac{p}{u-p} + l(u-p) = c,$$

ou

$$\frac{x(u-p)}{e^{-\frac{p}{u-p}}} = c,$$

c'est-à-dire

$$y - px = ce^{\frac{px}{y-px}}. \tag{6}$$

3^e^ Cas.—L'équation

$$u^2 - \frac{a}{n}u + \frac{ma}{n} + 1 = 0$$

a ses racines imaginaires. On peut alors mettre le trinôme du premier membre sous la forme

$$(u+a)^2 + b^2$$

et l'équation (4) devient

$$\frac{dx}{x} + \frac{udu}{(u+a)^2+b^2} = 0,$$

dont l'intégrale est

$$lx + \frac{1}{2}l\left[(u+a)^2+b^2\right] - \frac{a}{b}\,\text{arc tang}\,\frac{u+a}{b} = c,$$

ou

$$x\frac{\sqrt[2]{(u+a)^2+b^2}}{e^{\frac{a}{b}\,\text{arc tang}\,\frac{u+a}{b}}} = c,$$

ou encore

$$(y+ax)^2 + b^2x^2 = ce^{\frac{a}{b}\,\text{arc tang}\,\frac{y+ax}{bx}}. \tag{7}$$

Les équations (6) et (7) sont trop compliquées pour se prêter à une discussion intéressante.

6. Supposons maintenant que la courbe n qu'on se donne soit un cercle

$$(x_1-a)^2 + (y_1-b)^2 = r^2. \tag{8}$$

Portant dans cette équation les valeurs de x_1 et y_1 calculées plus haut, on a pour l'équation différentielle des courbes c correspondantes,

$$\left[\alpha xdx - a(xdx+ydy)\right]^2 + \left[\alpha ydx - b(xdx+ydy)\right]^2 = r^2(xdx+ydy)^2,$$

équation homogène du second degré que nous écrirons en posant pour simplifier $a^2+b^2-r^2=\lambda^2$,

$$\lambda^2\frac{y^2}{x^2}\left(\frac{dy}{dx}\right)^2 - 2\left[\alpha b\frac{y^2}{x^2} + (\alpha a-\lambda^2)\frac{y}{x}\right]\frac{dy}{dx} + \alpha^2\frac{y^2}{x^2} - 2\alpha b\frac{y}{x} + \alpha^2 - 2a\alpha + \lambda^2 = 0,$$

et où on opère la séparation des variables par la substitution $y=ux$, ce qui donne

$$\left.\begin{aligned}\lambda^2u^2\left(u+x\frac{du}{dx}\right)^2 - 2\left(\alpha bu^2 + (\alpha a-\lambda^2)u\right)\left(u+x\frac{du}{dx}\right)&\\ + \alpha^2u^2 - 2\alpha bu + \alpha^2 - 2a\alpha + \lambda^2 = 0.&\end{aligned}\right\} \tag{9}$$

De là on tire, après réduction,

$$u + x\frac{du}{dx} = \frac{abu + (a\alpha - \lambda^2) \pm \alpha\sqrt{(b^2 - \lambda^2)u^2 + 2abu + a^2 - \lambda^2}}{\lambda^2 u}$$

ou

$$\frac{dx}{x} + \frac{\lambda^2 u\,du}{\lambda^2 u^2 - abu - (a\alpha - \lambda^2) \mp \alpha\sqrt{(b^2 - \lambda^2)u^2 + 2abu + a^2 - \lambda^2}} = 0. \qquad (10)$$

On rendrait cette équation rationnelle par la substitution à la variable u de la variable z définie par

$$\sqrt{(b^2 - \lambda^2)u^2 + 2abu + a^2 - \lambda^2} = \sqrt{b^2 - \lambda^2}\,u + z;$$

mais on est ainsi conduit à des calculs d'une grande complication et sans intérêt véritable.

Nous nous bornerons à certains cas particuliers.

7. Supposons d'abord que le cercle ait son centre au point O. Alors

$$a = 0,\ b = 0,\ \lambda^2 = -r^2,$$

et l'équation (10) devient

$$\frac{dx}{x} + \frac{r u\,du}{r(u^2 + 1) \pm \alpha\sqrt{u^2 + 1}} = 0.$$

Posons

$$\sqrt{1 + u^2} = z,$$

d'où

$$1 + u^2 = z^2,\quad u\,du = z\,dz.$$

L'équation précédente se transforme alors ainsi

$$\frac{dx}{x} + \frac{r\,dz}{rz \pm \alpha} = 0.$$

L'intégrale de celle-ci est

$$x(rz \pm \alpha) = c$$

ou

$$x\left(r\sqrt{1 + \frac{y^2}{x^2}} \pm \alpha\right) = c,$$

ou encore

$$x^2 + y^2 = \frac{(c \pm \alpha x)^2}{r^2}.$$

On trouve ainsi deux coniques ayant O pour foyer et Ox pour axe. Ces coniques sont des ellipses, des paraboles ou des hyperboles suivant que α est inférieur, égal ou supérieur à r.

De là ce théorème :

Le lieu des points de rencontre des rayons vecteurs d'une conique issus d'un foyer et des parallèles aux normales correspondantes, menées par un point de l'axe focal, est un cercle ayant ce foyer pour centre.

Appliquant à ce cas particulier la construction générale du centre de courbure donnée au No. 2, on obtient ce théorème connu:

Si la normale en un point M d'une conique de foyer O coupe l'axe focal au point M' et que la perpendiculaire élevée en M' à MM' coupe le vecteur OM au point L, la perpendiculaire élevée en L à OM passe par le centre de courbure Ω relatif au point M.

8. Supposons maintenant que le cercle n passe par le point O et ait son centre sur l'axe Ox. Dans ce cas

$$\lambda = 0,\ b = 0,$$

et l'équation (9) devient

$$-2a\alpha u\left(u + x\frac{du}{dx}\right) + \alpha^2u^2 + \alpha^2 - 2a\alpha = 0,$$

ou, toute réduction faite,

$$\frac{dx}{x} = \frac{2audu}{(\alpha - 2a)(1 + u^2)}.$$

L'intégrale de cette équation est

$$lx = \frac{a}{\alpha - 2a}\, l(1 + u^2) + c,$$

ou

$$x = c(1 + u^2)^{\frac{a}{\alpha - 2a}},$$

ou encore

$$x = c(x^2 + y^2)^{\frac{a}{\alpha}}.$$

En posant

$$\frac{a}{\alpha} = \frac{m-1}{2m} \tag{11}$$

on voit que cette équation peut s'écrire

$$\sqrt{x^2 + y^2}\left(\frac{x}{\sqrt{x^2 + y^2}}\right)^m = c,$$

ou, en passant aux coordonnées polaires,

$$\rho \cos^m\omega = c.$$

Cette forme d'équation met bien en évidence la nature de ces courbes. Elles coupent normalement l'axe Ox au point situé à la distance $\rho = c$ de l'origine.

Si m est positif la courbe est parabolique.

Si m est négatif elle est fermée et passe par l'origine.

De (11) on tire

$$m = \frac{\alpha}{\alpha - 2a}, \tag{12}$$

m sera donc positif ou négatif selon que P sera extérieur ou intérieur au cercle n. Il en résulte, d'après ce qui a été dit au No. 3, que lorsque m est positif, c'est-à-dire lorsque la courbe est parabolique, il y a deux points d'inflexion.

Ces points d'inflexion sont évidemment symétriques par rapport à Ox. Soit θ l'inclinaison de leurs vecteurs sur Ox.

L'angle θ s'obtient en menant du point P la tangente PH au cercle n et irant OH (Fig. 2).

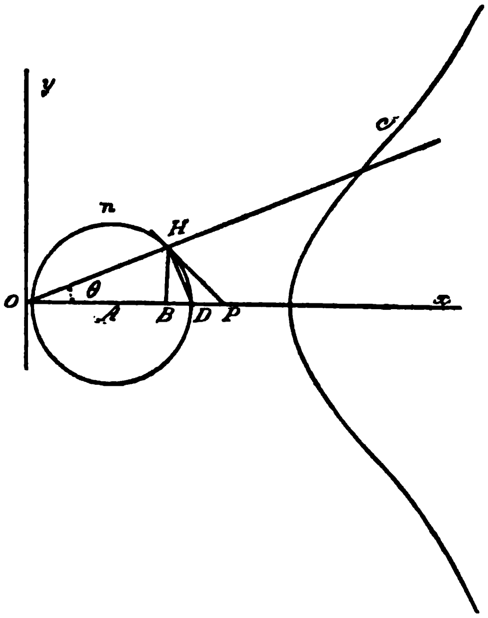

Fig. 2.

Soient A le centre du cercle n, $(OA = a)$, B le pied de la perpendiculaire abaissée de H sur OP. On a

$$\text{tang}^2\,\theta = \frac{HB^2}{OB^2} = \frac{OH^2 - OB^2}{OB^2} = \frac{OB.OD - OB^2}{OB^2} = \frac{OD - OB}{OB} = \frac{BD}{OB} = \frac{a - AB}{a + AB}.$$

Mais

$$AB = \frac{AD^2}{AP} = \frac{a^2}{\alpha - a}.$$

Donc

$$\text{tang}^2\,\theta = \frac{a - \dfrac{a^2}{\alpha - a}}{a + \dfrac{a^2}{\alpha - a}} = \frac{\alpha - 2a}{a},$$

ou, d'après (12),

$$\text{tang}^2\,\theta = \frac{1}{m}. \tag{13}$$

9. Revenons à la définition des courbes n (No. 1) et supposons maintenant que dans cette définition, nous remplacions les parallèles menées par le point P aux normales de la courbe c par des parallèles à ses tangentes, nous obtiendrons ainsi pour chaque position des points O et P une courbe que nous désignerons par la lettre t.

Les courbes t permettent de résoudre relativement à la courbe c les mêmes questions que les courbes n.

On voit immédiatement qu'elles passent par les points de contact des tangentes menées du point P à la courbe c, par les points communs aux normales à cette courbe issues de O et au cercle décrit sur OP comme diamètre, et que leurs asymptotes sont parallèles aux asymptotes de la courbe c et aux tangentes menées de O à cette courbe.

Elles permettent également de construire la tangente en un point de la courbe c et les points de contact des tangentes à cette courbe, parallèles à une direction donnée.

10. Elles permettent aussi, comme les courbes n, de déterminer le centre de courbure pour tout point de la courbe c.

Appelant H le point de la courbe t correspondant au point M de la courbe c (Fig. 3), reprenons toutes les notations du No. 2. Nous aurons encore ici par le même calcul

$$\frac{M\Omega}{MN} = \frac{HE}{HI}.$$

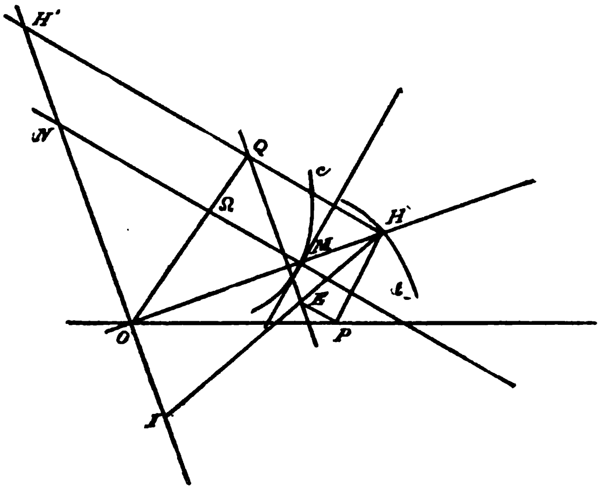

Fig. 3.

Nous mènerons aussi par le point E une perpendiculaire à OM, mais au lieu de prendre son point de rencontre avec HP nous le prendrons avec la perpendiculaire élevée en H à HP et nous appellerons ce point de rencontre Q. On aura alors, en appelant H' le point où HQ coupe ON,

$$\frac{HQ}{HH'}=\frac{HE}{HI}=\frac{M\Omega}{MN},$$

et cela montre que le point Ω se trouve sur la droite OQ.

Ici, la simplification que nous avons tirée, dans le cas de la courbe n, du théorème des trois hauteurs, ne se produit plus. Donc, la construction du centre de courbure est plus simple au moyen de la courbe n qu'au moyen de la courbe t; et c'est pourquoi nous avons d'abord considéré la courbe n.

11. Voici d'ailleurs comment le cas de la courbe t se ramène à celui de la courbe n (Fig. 4):

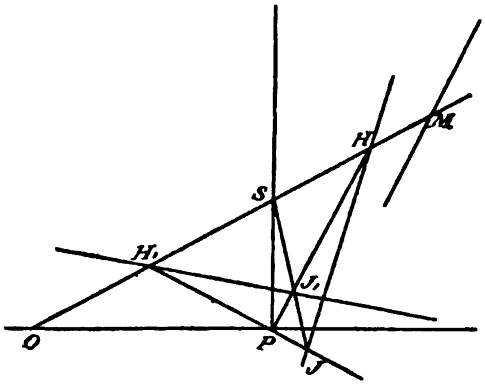

FIG. 4.

Soit H le point de la courbe t qui répond au point M de la courbe c. En P élevons à PH une perpendiculaire qui coupe OM en H_1; H_1 est le point correspondant de la courbe n. Pour appliquer la construction simple donnée à la fin du No. 2, il suffirait de connaître la tangente en H_1 à la courbe n; or cette tangente se déduit très aisément de la tangente en H à la courbe t, qui est ici l'élément connu, grâce à un théorème que nous avons donné ailleurs.*

Voici quelle est cette construction:

La droite OH coupe la perpendiculaire élevée en P à OP, qui est une droite fixe, en un point S. La tangente en H à la courbe t coupant PH_1 en J, on tire JS qui coupe PH en J_1. H_1J_1 est la tangente à la courbe n au point H_1.

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* *Journal de Mathématiques spéciales*, 1885, p. 15 et 33.

Si donc PH est tangente à t, PH_1 sera tangente à n. Rapprochant ce résultat du théorème donné au No. 3, on voit que les points de la courbe t correspondant aux points d'inflexion de la courbe c sont les points de contact des tangentes menées du point P à t.

12. Il nous reste à traiter pour les courbes t les mêmes problèmes que pour les courbes n.

Supposons d'abord que, se donnant la courbe t absolument quelconque, on cherche les courbes c correspondantes.

Prenons pour origine le point O, pour axe Ox la droite OP. Posons $OP = a$ et appelons x et y les coordonnées du point M, x_1 et y_1 les coordonnées du point H. Nous avons

$$\frac{y}{x} = \frac{y_1}{x_1},$$

$$\frac{dy}{dx} = \frac{y_1}{x_1 - a},$$

d'où

$$\left.\begin{aligned} x_1 &= \frac{axdy}{xdy - ydx}, \\ y_1 &= \frac{aydy}{xdy - ydx}. \end{aligned}\right\} \tag{14}$$

Portant ces valeurs de x_1 et y_1 dans l'équation de la courbe t donnée, on obtient l'équation différentielle des courbes c correspondantes.

13. Supposons d'abord que la courbe t donnée soit une droite

$$y_1 = mx_1 + n.$$

L'équation différentielle correspondante sera, d'après les formules (14),

$$aydy = maxdy + n(xdy - ydx)$$

ou

$$[(ma + n)x - ay]\,dy - nydx = 0,$$

équation homogène dans laquelle on sépare les variables par la substitution $y = ux$. On obtient alors

$$\frac{dx}{x} + \frac{[ma + n - au]\,du}{au(m - u)} = 0,$$

ou

$$\frac{dx}{x} + \frac{ma + n}{ma}\left[\frac{du}{u} - \frac{du}{u - m}\right] + \frac{du}{u - m} = 0,$$

dont l'intégrale est

$$x\left[\frac{u}{u-m}\right]^{\frac{m\alpha+n}{m\alpha}}(u-m)=c,$$

ou

$$y^{m\alpha+n}=c\,(y-mx)^{n}. \tag{15}$$

Lorsque m, n et α sont commensurables, cette équation peut toujours se mettre sous la forme

$$y^{p}=c\,(y-mx)^{q},$$

p et q étant des nombres entiers.

Pour $m\alpha+n=2n$, ou $m\alpha=n$, on a une parabole

$$y^{2}=c\,(y-mx),$$

qui passe par l'origine où elle est tangente à la droite $y-mx=0$. En outre, puisque $m\alpha=n$, le point P est symétrique par rapport à l'origine du point où la droite t coupe l'axe des x; en d'autres termes le point P est le pôle de la droite t; de là ce théorème:

Soient P un point quelconque pris dans le plan d'une parabole, O l'extrémité du diamètre de cette parabole qui passe par le point P. Le lieu du point de rencontre des vecteurs des points de la parabole, issus de O, et des parallèles aux tangentes correspondantes menées par le point P est la polaire du point P par rapport à la parabole.

La transformation homographique de ce théorème conduit à cet autre: *Soient P un point quelconque pris dans le plan d'une conique tangente en A à la droite a, O le second point où la droite PA coupe la conique. Si la tangente en un point M de la conique coupe la droite a au point T, les droites PT et OM se coupent sur la polaire du point P relativement à la conique.*

La propriété corrélative peut s'énoncer ainsi:

Soient p une droite quelconque située dans le plan d'une conique tangente en A à la droite a; du point commun aux droites p et a, on peut mener à la conique une seconde tangente d. Si la tangente en un point M quelconque de la conique coupe la droite d au point T et que la droite AM coupe la droite p au point I, la droite IT passe par le pôle de la droite p relativement à la conique.

13. A l'aide des formules (14) on formerait de même l'équation différentielle des courbes c ayant pour courbe t relativement aux points O et P un cercle donné. Mais on tomberait alors, ainsi qu'au No. 6, sur une équation homogène compliquée. Nous nous contenterons de traiter un cas particulier, celui où le cercle t a pour centre le point O.

Dans ce cas l'équation différentielle est

$$a^2x^2dy^2 + a^2y^2dy^2 = r^2(xdy - ydx)^2,$$

ou

$$[(a^2 - r^2)x^2 + a^2y^2]\,dy^2 + 2r^2xydxdy - r^2y^2dx^2 = 0.$$

Par la substitution $y = ux$, nous donnons à cette équation la forme

$$(a^2 - r^2 + a^2u^2)\,x^2(udx + xdu)^2 + 2r^2ux^2dx\,(udx + xdu) - r^2u^2x^2dx^2 = 0,$$

d'où

$$udx + xdu = \frac{-r^2u \pm aru\sqrt{1+u}}{a^2 - r^2 + a^2u^2}\,dx,$$

qu'on peut écrire

$$\frac{dx}{x} + \frac{(a^2u^2 + a^2 - r^2)\,du}{au\,[a(1+u^2) \mp r\sqrt{1+u^2}]} = 0,$$

ou

$$\frac{dx}{x} + \frac{audu}{a(1+u^2) \mp r\sqrt{1+u^2}} + \frac{a^2 - r^2}{a} \cdot \frac{du}{u\,[a(1+u^2) \mp r\sqrt{1+u^2}]} = 0.$$

Opérons le changement de variable

$$1 + u^2 = z^2,$$

d'où

$$udu = zdz.$$

Il vient

$$\frac{dx}{x} + \frac{adz}{az \mp r} + \frac{a^2 - r^2}{a}\,\frac{dz}{(z^2-1)(az \mp r)} = 0.$$

Effectuant la décomposition de la fraction rationnele qui figure au dernier terme on a

$$\frac{1}{(z^2-1)(az \mp r)} = \frac{1}{2(a \mp r)(z-1)} + \frac{1}{2(a \pm r)(z+1)} + \frac{1}{\frac{r^2 - a^2}{a}(az \mp r)}.$$

L'équation différentielle précédente devient donc

$$\frac{dx}{x} + \frac{adz}{az \mp r} + \frac{(a \pm r)\,dz}{2a(z-1)} + \frac{(a \mp r)\,dz}{2a(z+1)} - \frac{adz}{az \mp r} = 0,$$

ou

$$2a\frac{dx}{x} + (a \pm r)\frac{dz}{z-1} + (a \mp r)\frac{dz}{z+1} = 0,$$

dont l'intégrale est

$$x^{2a}(z-1)^{a \pm r}(z+1)^{a \mp r} = c,$$

ou

$$(\sqrt{x^2+y^2} - x)^{a \pm r}(\sqrt{x^2+y^2} + x)^{a \mp r} = c,$$

ou encore

$$y^{2a}\left(\frac{\sqrt{x^2+y^2} - x}{\sqrt{x^2+y^2} + x}\right)^{\pm r} = c,$$

qu'on peut aussi écrire en multipliant haut et bas par $(\sqrt{x^2+y^2}-x)^{\pm r}$,

$$y^{2(\alpha\mp r)}(\sqrt{x^2+y^2}-x)^{\pm 2r}=c,$$

ou

$$c^2y^{2\frac{\alpha}{\pm r}}+2cxy^{\frac{\mp r+\alpha}{\pm r}}=1.$$

Pour les doubles signes de toutes les formules précédentes il est bien évident que tous les signes supérieurs doivent être pris ensemble, ou tous les signes inférieurs ensemble.

Pour $\alpha=r$, auquel cas le cercle t a pour rayon OP, on a les deux courbes

$$c^2+2cx=y^2,$$
$$c^2y^2+2cx=1,$$

paraboles d'axe Ox et de foyer O, résultat qui pouvait être prévu *à priori*. On a ainsi une vérification de la solution précédente.

14. Si on se reporte à la Fig. 3, on voit que le point M peut être considéré à chaque instant comme ayant une vitesse dont les composantes sont les parallèles à OH et à PO menées par le point M. La tangente à la courbe que décrit le point M est, en effet, par définition, parallèle à PH.

PO est fixe. En outre, pour le cas traité au No. 13, OH est constant puisque le lieu de H est un cercle de centre O. Le point M peut donc être alors considéré comme ayant une vitesse dont la composante parallèle à OP et la composante dirigée vers le point O sont constantes.

La courbe lieu du point M est donc *celle que décrit un point se dirigeant vers un point fixe avec une vitesse constante tout en étant entrainé par un courant de direction constante et de vitesse uniforme.*

On pourrait l'appeler la *courbe du nageur* car c'est celle que parcourt un nageur cherchant à atteindre un point fixe du rivage.

Cette courbe a déjà été rencontrée par M. Collignon.*

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* *Association française.* Congrès de Toulouse, 1887.

On the Surfaces with Plane or Spherical Curves of Curvature.

BY PROF. CAYLEY.

The theory is considered in two nearly cotemporaneous papers—Bonnet, "Mémoire sur les surfaces dont les lignes de courbure sont planes ou sphériques," *Jour. de l'Ecole Polyt.* t. XX (1853), pp. 117–306, and Serret, "Mémoire sur les surfaces dont toutes les lignes de courbure sont planes ou sphériques," *Liouville,* t. XVIII (1853), pp. 113–162. I desire to reproduce in a more compact form, and with some additional developments, the chief results obtained in these elaborate memoirs.

The basis of the theory is a theorem by Lancret, 1806. In any curve described upon a surface, the angle between the osculating planes at consecutive points is equal to the difference of the angles between the osculating planes and the corresponding tangent planes of the surface.

This includes as a particular case Joachimsthal's theorem, *Crelle,* t. XXX (1846): If a surface have a plane curve of curvature, then at any point thereof the angle between the plane of the curve and the tangent plane of the surface has a constant value.

Bonnet and Serret each deduce the like theorem for a spherical curve of curvature, viz: If a surface have a spherical curve of curvature, then at any point thereof the angle between the tangent plane of the sphere and the tangent plane of the surface has a constant value. Bonnet (Mémoire, p. 235) says that this follows from Lancret's theorem. Serret (Mémoire, p. 128) obtains it, by the transformation by reciprocal radius vectors, from Joachimsthal's theorem.

I remark that the theorem for a spherical curve of curvature, and (as a particular case thereof) that for a plane curve of curvature, are obtained at once from the most elementary geometrical considerations, viz: if we have (in the same plane or in different planes) the two isosceles triangles NPP', OPP' on a common base PP', then the angle OPN is equal to the angle $OP'N$. For take

P, P' consecutive points on a spherical curve of curvature; then at P, P' the normals of the surface meet in a point N, and the normals (or radii) of the sphere meet in the centre O, and we have angle $OPN =$ angle $OP'N$, that is, at each of these points the inclination of the normal of the surface to the normal of the sphere has the same value; and this value being thus the same for any two consecutive points, must be the same for all points of the curve of curvature. The proof applies to the plane curve of curvature; but in this case the fundamental theorem may be taken to be, a line at right angles to the base PP' of the isosceles triangle NPP' is equally inclined to the two equal sides NP, NP'.

A surface may have one set of its curves of curvature plane or spherical. To include the two cases in a common formula, the equation may be written $k(x^2+y^2+z^2) - 2ax - 2by - 2cz - 2u = 0$; $k = 1$ in the case of a sphere, $= 0$ in that of a plane; and the expression a sphere may be understood to include a plane. I write in general A, B, C to denote the cosines of the inclinations of the normal of the surface at the point (x, y, z) to the axes of coordinates (consequently $A^2 + B^2 + C^2 = 1$). Hence considering a surface, and writing down the equations

$$k(x^2+y^2+z^2) - 2ax - 2by - 2cz - 2u = 0,$$
$$(kx-a)A + (ky-b)B + (kz-c)C = l,$$

where (a, b, c, u, l) are regarded as functions of a parameter t, the first of these equations is that of a variable sphere; and the second equation expresses that at a point of intersection of the surface with the sphere, the inclination of the tangent plane of the surface to the tangent plane of the sphere has a constant value l, viz: this is a value depending only on the parameter t, and therefore constant for all points of the curve of intersection of the sphere and surface: by what precedes, the curve of intersection is a curve of curvature of the surface, and the surface will thus have a set of spherical curves of curvature.

Supposing the surface defined by means of expressions of its coordinates (x, y, z) as functions of two variable parameters, we may for one of these take the parameter t which enters into the equation of the sphere; and if the other parameter be called θ, then the expressions of the coordinates are of the form $x, y, z = x(t, \theta), y(t, \theta), z(t, \theta)$ respectively; these give equations $dx, dy, dz = a\,dt + a'd\theta,\ b\,dt + b'd\theta,\ c\,dt + c'd\theta$ (where of course (a, b, c, a', b', c') are in general functions of t, θ), and we have A, B, C proportional to

$bc' - b'c$, $ca' - c'a$, $ab' - a'b$, viz: the values are equal to these expressions each divided by the square root of the sum of their squares. In order that the surface may have a set of spherical curves of curvature, the above three equations must be satisfied identically by means of the values of

$$a, b, c, u, l, A, B, C, x, y, z$$

as functions of (t, θ); and it may be seen without difficulty that we are thereby led to a partial differential equation of the first order for the determination of the surface. But I do not at present further consider this question of the determination of a surface having one set of its curves of curvature (plane or) spherical.

Suppose now that there is a second set of (plane or) spherical curves of curvature. We have in like manner

$$\begin{aligned} \varkappa(x^2 + y^2 + z^2) - 2\alpha x - 2\beta y - 2\gamma z - 2v &= 0, \\ (\varkappa x - \alpha)A + (\varkappa y - \beta)B + (\varkappa z - \gamma)C - \lambda &= 0, \end{aligned}$$

where $\varkappa$ is $=1$ or $=0$ according as the curves are spherical or plane, and $(\alpha, \beta, \gamma, v, \lambda)$ are functions of a variable parameter θ. We take the t of the former set of equations and the θ of these equations as the two parameters in terms of which the coordinates (x, y, z) are expressed. This being so (the former equations being satisfied as before), if these equations are satisfied identically by the values of $\alpha, \beta, \gamma, v, \lambda, A, B, C, x, y, z$ as functions of (t, θ), then the surface will have its other set of curves of curvature also spherical. It will be recollected that by hypothesis a, b, c, u, l are functions of the parameter t only, and that $\alpha, \beta, \gamma, v, \lambda$ functions of the parameter θ only. The foregoing equations, together with the assumed relations

$$\begin{aligned} A^2 + B^2 + C^2 &= 1, \\ Adx + Bdy + Cdz &= 0, \end{aligned}$$

are the "six equations" for the determination of a surface having its two sets of curves of curvature each of them (plane or) spherical.

Assuming now the values of a, b, c, l, u as functions of t, and $\alpha, \beta, \gamma, \lambda, v$ as functions of θ, the question at once arises whether we can then satisfy the six equations. These equations other than $Adx + Bdy + Cdz = 0$, or say the five equations, in effect determine any five of the eight quantities $A, B, C,$

x, y, z, t, θ, in terms of the remaining three, say they determine A, B, C, t, θ as functions of x, y, z: we thus have a differential equation $Adx + Bdy + Cdz = 0$ wherein A, B, C are to be regarded as given functions of (x, y, z). An equation of this form is not in general integrable; and if the equation in question be not integrable, then clearly the system of equations cannot be satisfied by any value of z as a function of (x, y), or, what is the same thing, by any values of (x, y, z) as functions of (t, θ). We thus arrive at the condition, the equation must be integrable, viz: the condition is

$$\nabla, = A\left(\frac{dB}{dz} - \frac{dC}{dy}\right) + B\left(\frac{dC}{dx} - \frac{dA}{dz}\right) + C\left(\frac{dA}{dy} - \frac{dB}{dx}\right), = 0.$$

If this be satisfied, then we have an integral equation $I = 0$ (containing a constant of integration which is an absolute constant) and which is in fact the equation of the required surface. But it is proper to look at the question somewhat differently. Supposing that the condition $\nabla = 0$ is satisfied, then we have the integral equation $I = 0$, and this equation, together with the five equations, in effect determine any six of the quantities $A, B, C, x, y, z, t, \theta$ in terms of the remaining two of them, or, what is the same thing, they determine a relation between any three of these quantities. We can, from the five equations and their differentials, and from the equation $Adx + Bdy + Cdz = 0$, obtain a differential equation between any three of the eight quantities: and it has just been seen that corresponding hereto we have an integral relation between the same three quantities; that is (the condition $\nabla = 0$ being satisfied), we can from the six equations obtain between any three of the quantities $A, B, C, x, y, z, t, \theta$ a linear differential equation of the foregoing form (for instance $Zdz + Tdt + \Theta d\theta = 0$, where Z, T, Θ are given functions of z, t, θ) which will *ipso facto* be integrable, furnishing between z, t, θ an integral equation which may be used instead of the before-mentioned integral equation $I = 0$. And we thus have (without any further integration) in all six equations which serve to determine any six of the quantities $A, B, C, x, y, z, t, \theta$ in terms of the remaining two. It is often convenient to seek in this way for the expressions of (A, B, C and) x, y, z as functions of t, θ in preference to seeking for the integral equation $I = 0$ between the coordinates x, y, z.

The condition $\nabla = 0$ is in fact the condition which expresses that at any point of the surface the two curves of curvature intersect at right angles. Serret (and after him Bonnet) in effect obtain the condition by the assumption of this geometrical relation, without showing that the geometrical relation is in fact the necessary condition for the coexistence of the six equations. They give the condition in the form $dx\delta x + dy\delta y + dz\delta z = 0$, where dx, dy, dz are the increments of (x, y, z) along one of the curves of curvature, and δx, δy, δz the increments along the other curve of curvature. The equations give

$$\begin{aligned} (kx-a)\,dx + (ky-b)\,dy + (kz-c)\,dz &= 0\,, \\ A dx + B dy + C dz &= 0\,, \end{aligned}$$

and similarly

$$\begin{aligned} (\kappa x-\alpha)\,\delta x + (\kappa y-\beta)\,\delta y + (\kappa z-\gamma)\,\delta z &= 0\,, \\ A\delta x + B\delta y + C\delta z &= 0\,. \end{aligned}$$

We thence have

$$\begin{gathered} dx : dy : dz = \\ B(kz-c) - C(ky-b) : C(kx-a) - A(kz-c) : A(ky-b) - B(kx-a)\,, \end{gathered}$$

and

$$\begin{gathered} \delta x : \delta y : \delta z = \\ B(\kappa z-\gamma) - C(\kappa y-\beta) : C(\kappa x-\alpha) - A(\kappa z-\gamma) : A(\kappa y-\beta) - B(\kappa x-\alpha)\,. \end{gathered}$$

We have thus the required condition, in a form which is readily changed into

$$\begin{gathered} (A^2+B^2+C^2)\{(kx-a)(\kappa x-\alpha) + (ky-b)(\kappa y-\beta) + (kz-c)(\kappa z-\gamma)\} \\ -\{A(kx-a)+B(ky-b)+C(kz-c)\}\{A(\kappa x-\alpha)+B(\kappa y-\beta)+C(\kappa z-\gamma)\} = 0\,, \end{gathered}$$

and writing herein $A^2+B^2+C^2=1$, this becomes

$$\begin{aligned} &\tfrac{1}{2}\kappa\,\{k(x^2+y^2+z^2) - 2ax - 2by - 2cz\} \\ &+ \tfrac{1}{2}k\,\{\kappa(x^2+y^2+z^2) - 2\alpha x - 2\beta y - 2\gamma z\} \\ &+ (a\alpha + b\beta + c\gamma) - l\lambda = 0\,, \end{aligned}$$

that is,

$$a\alpha + b\beta + c\gamma - l\lambda + \kappa u + kv = 0\,.$$

I proceed to show that this is the condition $\nabla = 0$ for the integrability of the differential equation $Adx + Bdy + Cdz = 0$. Writing as before

$$\nabla = A\left(\frac{dB}{dz} - \frac{dC}{dy}\right) + B\left(\frac{dC}{dx} - \frac{dA}{dz}\right) + C\left(\frac{dA}{dy} - \frac{dB}{dx}\right),$$

we have from the six equations

$$\begin{aligned}
&AdA+BdB+CdC=0,\\
&(kx-a)\,dA+(ky-b)\,dB+(kz-c)\,dC\\
&\qquad\qquad=-k\,(Adx+Bdy+Cdz)+(Aa_1+Bb_1+Cc_1+l_1)\,dt,\\
&(\varkappa x-\alpha)\,dA+(\varkappa y-\beta)\,dB+(\varkappa z-\gamma)\,dC\\
&\qquad\qquad=-\varkappa\,(Adx+Bdy+Cdz)+(A\alpha'+B\beta'+C\gamma'+\lambda')\,d\theta,\\
&(kx-a)\,dx+(ky-b)\,dy+(kz-c)\,dz=(a_1x+b_1y+c_1z+u_1)\,dt,\\
&(\varkappa x-\alpha)\,dx+(\varkappa y-\beta)\,dy+(\varkappa z-\gamma)\,dz=(\alpha'x+\beta'y+\gamma'z+\upsilon')\,d\theta,
\end{aligned}$$

where a_1, b_1, c_1, l_1, u_1 denote derived functions in regard to t and $\alpha', \beta', \gamma', \lambda', \upsilon'$ derived functions in regard to θ. Putting for shortness

$$\Omega=\begin{vmatrix} A, & B, & C,\\ kx-a, & ky-b, & kz-c\\ \varkappa x-\alpha, & \varkappa y-\beta, & \varkappa z-\gamma \end{vmatrix}$$

we readily obtain

$$\begin{aligned}
\Omega dA=\;&[(\varkappa y-\beta)\,C-(\varkappa z-\gamma)\,B]\Big\{-k\,(Adx+Bdy+Cdz)\\
&\qquad+\frac{Aa_1+Bb_1+Cc_1+l_1}{a_1x+b_1y+c_1z+u_1}\{(kx-a)\,dx+(ky-b)\,dy+(kz-c)\,dz\}\Big\}\\
-\;&[(ky-b)\,C-(kz-c)\,B]\Big\{-\varkappa\,(Adx+Bdy+Cdz)\\
&\qquad+\frac{A\alpha'+B\beta'+C\gamma'+\lambda'}{\alpha'x+\beta'y+\gamma'z+\upsilon'}\{\varkappa x-\alpha)\,dx+(\varkappa y-\beta)\,dy+(\varkappa z-\gamma)\,dz\}\Big\};
\end{aligned}$$

say this is

$$\begin{aligned}
\Omega dA=\;&[(\varkappa y-\beta)\,C-(\varkappa z-\gamma)\,B]\Big\{-k\,(Adx+Bdy+Cdz)\\
&\qquad+\frac{L}{P}\{(kx-a)\,dx+(ky-b)\,dy+(kz-c)\,dz\}\Big\}\\
-\;&[(ky-b)\,C-(kz-c)\,B]\Big\{-\varkappa\,(Adx+Bdy+Cdz)\\
&\qquad+\frac{\Lambda}{\Pi}\{(\varkappa x-\alpha)\,dx+(\varkappa y-\beta)\,dy+(\varkappa z-\gamma)\,dz\}\Big\},
\end{aligned}$$

or introducing further abbreviations, and writing down the analogous values of ΩdB and ΩdC, we have

$$\begin{aligned}
\Omega dA&=[(\varkappa y-\beta)\,C-(\varkappa z-\gamma)\,B]\,U-[(ky-b)\,C-(kz-c)\,B]\,\Upsilon,\\
\Omega dB&=[(\varkappa z-\gamma)\,A-(\varkappa x-\alpha)\,C]\,U-[(kz-c)\,A-(kx-a)\,C]\,\Upsilon,\\
\Omega dC&=[(\varkappa z-\alpha)\,B-(\varkappa y-\beta)\,A]\,U-[(kx-a)\,B-(ky-b)\,A]\,\Upsilon.
\end{aligned}$$

We hence find

$$\begin{aligned}\Omega\frac{dB}{dz} = {} & [(\varkappa z-\gamma)A-(\varkappa x-\alpha)C]\left(-kC+\frac{L}{P}(kz-c)\right)\\ & -[(kz-c)A-(kx-a)C]\left(-\varkappa C+\frac{\Lambda}{\Pi}(\varkappa z-\gamma)\right),\end{aligned}$$

$$\begin{aligned}-\Omega\frac{dC}{dy} = {} & -[(\varkappa x-\alpha)B-(\varkappa y-\beta)A]\left(-kB+\frac{L}{P}(ky-b)\right)\\ & +[(kx-a)B-(ky-b)A]\left(-\varkappa B+\frac{\Lambda}{\Pi}(\varkappa y-\beta)\right),\end{aligned}$$

and combining these two terms, in the resulting value of $\Omega\left(\frac{dB}{dz}-\frac{dC}{dy}\right)$ first the term without L or Λ is found to be

$$\begin{aligned}= & -kA\{A(\varkappa x-\alpha)+B(\varkappa y-\beta)+C(\varkappa z-\gamma)\}\\ & -k(\varkappa x-\alpha)(A^2+B^2+C^2)\\ & +\varkappa(kx-a)(A^2+B^2+C^2)\\ & +\varkappa A\{A(kx-a)+B(ky-b)+C(kz-c)\},\end{aligned}$$

which is

$$\begin{aligned}& = -kA\lambda+k(\varkappa x-\alpha)-\varkappa(kx-a)+\varkappa Al,\\ & = \quad A(\varkappa l-k\lambda)-ka+\varkappa a.\end{aligned}$$

Next the coefficient of $\frac{L}{P}$ is

$$\begin{aligned}& A(kz-c)(\varkappa z-\gamma)-C(\varkappa x-\alpha)(kz-c)\\ & +A(ky-b)(\varkappa y-\beta)-B(\varkappa x-\alpha)(ky-b),\end{aligned}$$

which is

$$\begin{aligned}& = A[(kx-a)(\varkappa x-\alpha)+(ky-b)(\varkappa y-\beta)+(kz-c)(\varkappa z-\gamma)]\\ & \qquad\qquad -(\varkappa x-\alpha)[A(kx-a)+B(ky-b)+C(kz-c)]\\ & = AM+(\varkappa x-\alpha)l,\end{aligned}$$

if for shortness

$$M=(kx-a)(\varkappa x-\alpha)+(ky-b)(\varkappa x-\beta)+(kz-c)(\varkappa z-\gamma);$$

and similarly the coefficient of $\frac{\Lambda}{\Pi}$ is

$$\begin{aligned}& -A(kz-c)(\varkappa z-\gamma)+C(kx-a)(\varkappa x-\alpha)\\ & -A(ky-b)(\varkappa y-\beta)+B(kx-b)(\varkappa y-\beta)\end{aligned}$$

which is

$$\begin{aligned}&= -A\,[(kx-a)(\kappa x-\alpha)+(ky-b)(\kappa y-\beta)+(kz-c)(\kappa z-\gamma)]\\ &\quad -(kx-a)[A\,(\kappa x-\alpha)+B\,(\kappa y-\beta)+C\,(\kappa z-\gamma)]\\ &= -AM-(kx-a)\,\lambda .\end{aligned}$$

We thus obtain

$$\Omega\left(\frac{dB}{dz}-\frac{dC}{dy}\right)=A\,(\kappa l-k\lambda)-k\alpha+\kappa a+\frac{L}{P}\{AM+(\kappa x-\alpha)\,l\}$$
$$-\frac{\Lambda}{\Pi}\{AM+(kx-a)\,\lambda\},$$

and similarly

$$\Omega\left(\frac{dB}{dx}-\frac{dA}{dz}\right)=B\,(\kappa l-k\lambda)-k\beta+\kappa b+\frac{L}{P}\,\{BM+(\kappa y-\beta)\,l\}$$
$$-\frac{\Lambda}{\Pi}\,\{BM+(ky-b)\,\lambda\},$$

$$\Omega\left(\frac{dA}{dy}-\frac{dB}{dx}\right)=C\,(\kappa l-k\lambda)-k\gamma+\kappa c+\frac{L}{P}\,\{CM+(\kappa z-\gamma)\,l\}$$
$$-\frac{\Lambda}{\Pi}\,\{CM+(kz-c)\,\lambda\},$$

and hence multiplying by A, B, C and adding, we obtain

$$\Omega\nabla=\kappa l-k\lambda-k\,(A\alpha+B\beta+C\gamma)+\kappa\,(Aa+Bb+Cc)$$
$$+\frac{L}{P}\,(M-l\lambda)-\frac{\Lambda}{\Pi}\,(M-l\lambda),$$

where the first four terms are together

$$=\kappa l-k\lambda+k\,\{\kappa\,(Ax+By+Cz)-\lambda\}-\kappa\,\{k\,(Ax+By+Cz)-l\},$$

viz. these destroy each other, and the equation becomes

$$\Omega\nabla=\left(\frac{L}{P}-\frac{\Lambda}{\Pi}\right)(M-l\lambda).$$

But we have

$$M-l\lambda=\tfrac{1}{2}\,\kappa\,\{k\,(x^2+y^2+z^2)-2ax-2by-2cz\}$$
$$+\tfrac{1}{2}\,k\,\{\kappa\,(x^2+y^2+z^2)-2\alpha x-2\beta y-2\gamma z\}+(a\alpha+b\beta+c\gamma)-l\lambda,$$

which is $\qquad = a\alpha+b\beta+c\gamma-l\lambda+\kappa u+kv,$

or we find $\qquad \Omega\nabla=\left(\dfrac{L}{P}-\dfrac{\Lambda}{\Pi}\right)(a\alpha+b\beta+c\gamma-l\lambda+\kappa u+kv),$

viz. the condition $\nabla=0$ is

$$a\alpha+b\beta+c\gamma-l\lambda+\kappa u+kv=0,$$

the result which was to be proved.

If we consider separately the cases where the two sets of curves of curvature are each plane, the first plane and the second spherical, and each of them spherical; or say the cases PP, PS and SS, then in these cases respectively the condition is

$$\begin{aligned} &a\alpha + b\beta + c\gamma - l\lambda = 0\,, \\ &a\alpha + b\beta + c\gamma - l\lambda + u = 0\,, \\ &a\alpha + b\beta + c\gamma - l\lambda + u + v = 0: \end{aligned}$$

we have in each case to take the italic letters functions of t and the greek letters functions of θ, satisfying identically the appropriate equation, but otherwise arbitrary; and then in each case the six equations lead to a differential equation $Adx + Bdy + Cdz = 0$ (or say $Zdz + Tdt + \Theta d\theta = 0$) between three variables, which equation is *ipso facto* integrable; and we thus obtain a new integral equation which with the original five integral equations gives the solution of the problem. The condition is in each case of the form $\Sigma a\alpha = 0$, the number of terms $a\alpha$ being 4, 5 or 6. Considering for instance the form

$$a\alpha + b\beta + c\gamma + d\delta + e\varepsilon + f\phi = 0$$

with 6 terms, it is easy to see how such an equation is to be satisfied by values of a, b, c, d, e, f which are functions of t, and values of α, β, γ, δ, ε, ϕ which are functions of θ. Suppose that t_1, t_2, ... are particular values of t, and a_1, b_1, ...f_1; a_2, b_2, ... f_2, etc., the corresponding values of a, b, ...f, these values being of course absolute constants; we have α, β, ... ϕ, functions of θ, satisfying all the equations

$$\begin{aligned} &(a_1, b_1, c_1, d_1, e_1, f_1)(\alpha, \beta, \gamma, \delta, \varepsilon, \phi) = 0\,, \\ &(a_2, b_2, c_2, d_2, e_2, f_2)(\qquad \text{"} \qquad) = 0\,, \\ &\text{etc.,} \end{aligned}$$

and if 6 or more of these equations were independent, the equations could, it is clear, be satisfied only by the values $\alpha = \beta = \gamma = \delta = \varepsilon = \phi = 0$. To obtain a proper solution, only some number less than 6 of these equations can be independent. Suppose for instance only two of the equations are independent; we then have α, β, γ, δ, ε, ϕ functions of θ satisfying these equations, but otherwise arbitrary; or, what is the same thing, we may take α, β, γ, δ, ε, ϕ linear functions of $6 - 2, = 4$ arbitrary functions, say P, Q, R, S of θ; say we have

$$\begin{aligned} &\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)(P, Q, R, S), \\ &\beta = (\beta_0, \ldots\ldots)(\qquad \text{"} \qquad), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\phi = (\phi_0, \ldots\ldots)(\qquad \text{"} \qquad), \end{aligned}$$

where the suffixed greek letters denote absolute constants; and this being so, in order to satisfy the proposed equation $a\alpha + b\beta + c\gamma + d\delta + e\varepsilon + f\phi = 0$, we must have

$$\begin{array}{l}
(\alpha_0, \beta_0, \gamma_0, \delta_0, \varepsilon_0, \phi_0)(a, b, c, d, e, f) = 0,\\
(\alpha_1, \ldots\ldots\ldots\ldots)(\qquad \text{"} \qquad) = 0,\\
(\alpha_2, \ldots\ldots\ldots\ldots)(\qquad \text{"} \qquad) = 0,\\
(\alpha_3, \ldots\ldots\ldots\ldots)(\qquad \text{"} \qquad) = 0,
\end{array}$$

viz. a, b, c, d, e, f will then be functions of t satisfying these four equations, but otherwise arbitrary. The above is a solution for the partition $2 + 4$ of the number 6. We have in like manner a solution for any other partition of 6; or if we disregard the extreme cases $a = b = c = d = e = f = 0$ and $\alpha = \beta = \gamma = \delta = \varepsilon = \phi = 0$, then we have in this manner solutions for the several partitions 15, 24, 33, 42 and 51 of the number 6.

But applying this theory to the actual problem, there is a good deal of difficulty as regards the enumeration of the really distinct cases. I use the letters P, S to denote that a set of curves of curvature is plane or spherical as the case may be, the surfaces to be considered are thus PP, PS, and SS. First, for the PP problem where the equation is $a\alpha + b\beta + c\gamma - l\lambda = 0$, the two systems (a, b, c, l) and $(\alpha, \beta, \gamma, \lambda)$ are symmetrically related to each other, and instead of the solutions 13, 22 and 31, it is sufficient to consider the solutions 13 and 22. But here (a, b, c, l) are not a system of four symmetrically related functions, (a, b, c) are a symmetrical system, and l is a distinct term: and the like for the system $(\alpha, \beta, \gamma, \lambda)$. In the PS problem, where the equation is $a\alpha + b\beta + c\gamma - l\lambda + u = 0$, and thus the systems (a, b, c, l, u), $(\alpha, \beta, \gamma, \lambda, 1)$ are of different forms, we should consider the solutions 14, 23, 32 and 41: but here again in each of the systems separately the terms are not symmetrically related to each other. Lastly, in the SS problem where the equation is $a\alpha + b\beta + c\gamma - l\lambda + u + v = 0$, the systems $(a, b, c, l, u, 1)$ and $(\alpha, \beta, \gamma, \lambda, 1, v)$ are of the same form, it is enough to consider the solutions 15, 24 and 33; but in this case also in each of the systems separately the terms are not symmetrically related to each other. I do not at present further consider the question, but simply adopt Serret's enumeration.

It is to be remarked that for a developable (but not for a skew surface) the generating lines may be curves of curvature, and regarding the generating lines as plane curves we might have developables PP or PS; but a straight line is not a curve in a *determinate* plane, and it is better to consider the case apart from the general theory. Again, the curves of curvature of one set or those of each set may be circles; and a circle may be regarded either as a plane or a spherical curve; regarding it, however, as a spherical curve, it is a curve not in a *determinate* sphere. The cases in question, of the curves of curvature of the one set or of those of each set being circles, are therefore also to be considered apart from the general theory. The surfaces referred to present themselves for consideration among Serret's cases PP 1°, 2°, 3°; PS 1°, 2°, 3°, 4°, 5°, 6°, 7°; and SS 1°, 2°, 3°, 4°; but they are excluded from his enumeration, and he in fact reckons in his "Conclusion," pp. 161, 162, two kinds of surfaces PP, three kinds PS, and two kinds SS.

It is very easily seen that if a surface has a plane or spherical curve of curvature, then on any parallel surface the corresponding curve is a plane or spherical curve of curvature: and thus if a surface be PP, PS, or SS, then the parallel surfaces are respectively PP, PS or SS. The solutions obtained include for the most part all the parallel surfaces, and thus there is no occasion to make use of this theorem; but see in the continuation of the present paper the case considered under the subheading *post*, $PS4^0 =$ Serret's third case of PS.

If a surface have a plane or spherical curve of curvature, then transforming the surface by reciprocal radius vectors (or inverting in regard to an arbitrary point), then in the transformed surface the corresponding curve is a spherical curve of curvature. Hence if a surface be PP, PS or SS, the transformed surface is SS. Conversely, as shown by Bonnet and Serret, and as will appear, every surface SS is in fact an inversion of a surface PP or PS.

I proceed to the enumeration, developing the theory only in regard to the two, three, and two, cases PP, PS and SS respectively.

PP, THE TWO SETS OF CURVES OF CURVATURE EACH PLANE.

The six equations are

$$\begin{aligned}
A^2 + B^2 + C^2 \quad &= 1, \\
ax + by + cz + u &= 0, \\
Aa + Bb + Cc + l &= 0, \\
\alpha x + \beta y + \gamma z + v &= 0, \\
A\alpha + B\beta + C\gamma + \lambda &= 0, \\
Adx + Bdy + Cdz \quad &= 0,
\end{aligned}$$

the condition is

$$a\alpha + b\beta + c\gamma - l\lambda = 0$$

(not containing u or v, so that these remain arbitrary functions of t, θ respectively). The cases are

	a	b	c	l	α	β	γ	λ
$PP1^0$	1	0	c	0	0	1	0	λ
$PP2^0$	0	1	0	$-m$	α	$-m\lambda$	γ	λ
$PP3^0$	1	0	c	mc	0	1	$m\lambda$	λ

m is an arbitrary constant, and in the body of the table c is an arbitrary function of t, and α, γ, λ arbitrary functions of θ.

$PP1^0$ is Serret's first case of PP, included in his second case.

$PP2^0$ gives developable.

$PP3^0$ is Serret's second case of PP.

I consider the case

$PP3^0$ = SERRET'S SECOND CASE OF PP.

Writing for greater symmetry $m = g$, $\dfrac{1}{m} = f$, so that $fg = 1$; also $m\lambda = \gamma$, and consequently $\lambda = f\gamma$, we take c and γ for the two parameters respectively, or write $c = t$, $\gamma = \theta$; also changing the letters u, v, we write

	a	b	c	l	u	α	β	γ	λ	v
=	1,	0,	t	gt	P	0	1	θ	$f\theta$	Π

and the six equations thus are

$$\begin{aligned}
A^2 + B^2 &+ C^2 &&= 1,\\
x + tz &- P &&= 0,\\
A + tC &- gt &&= 0,\\
y + \theta z &- \Pi &&= 0,\\
B + \theta C &- f\theta &&= 0,\\
Adx + Bdy &+ Cdz &&= 0.
\end{aligned}$$

We seek for the differential equation in z, t, θ. We have

$$A^2 + B^2 + C^2 = 1,\ \ A = t(g - C),\ \ B = \theta(f - C),$$

and thence $$t^2(g-C)^2+\theta^2(f-C)^2+C^2=1,$$

that is $$C^2(1+t^2+\theta^2)+2C(gt^2+f\theta^2)=1-g^2t^2-f^2\theta^2,$$

or multiplying by $1+t^2+\theta^2$ and completing the square,

$$\begin{aligned}\{(1+t^2+\theta^2)\,C-gt^2-f\theta^2\}^2 &= (1-g^2t^2-f^2\theta^2)(1+t^2+\theta^2)-(gt^2+f\theta^2)^2\\ &= \{f+(f-g)\,t^2\}\,\{g+(g+(g-f)\,\theta^2\}\\ &= \frac{1}{T^2\Theta^2},\end{aligned}$$

if

$$\frac{1}{T^2}=f+(f-g)\,t^2,$$

$$\frac{1}{\Theta^2}=g+(g-f)\,\theta^2,$$

and thence giving a determinate sign to the square root, say

$$(1+t^2+\theta^2)\,C=gt^2+f\theta^2-\frac{1}{T\Theta},$$

an equation which may also be written

$$C=\frac{fT-g\Theta}{f-g}.$$

In fact, observing that $\frac{1}{T^2}-\frac{1}{\Theta^2}=(f-g)(1+t^2+\theta^2)$, we deduce from the original form

$$\begin{aligned}\left(\frac{1}{T^2}-\frac{1}{\Theta^2}\right)C &= (f-g)(gt^2+f\theta^2)-\frac{f-g}{T\Theta},\\ &= g\left(\frac{1}{T^2}-f\right)-f\left(\frac{1}{\Theta^2}-g\right)-\frac{f-g}{T\Theta}\\ &= \left(\frac{g}{T}-\frac{f}{\Theta}\right)\left(\frac{1}{T}+\frac{1}{\Theta}\right),\end{aligned}$$

or throwing out the factor $\frac{1}{T}+\frac{1}{\Theta}$, and reducing, we have the required value; and thence forming the values of A and B, we have

$$A=-tT\frac{f-g}{T-\Theta},\quad B=-\theta\Theta\,\frac{f-g}{T-\Theta},\quad C=\frac{fT-g\Theta}{T-\Theta};$$

we have, moreover,

$$x+tz=P,\quad y+\theta z=\Pi,$$

or differentiating, and writing P_1 and Π' for the derived functions in regard to t and θ respectively,

$$dx=-tdz-zdt+P_1dt,\quad dy=-\theta dz-zd\theta-\Pi'd\theta.$$

The equation $Adx + Bdy + Cdz = 0$ thus becomes

$$- Ttdx - \Theta\theta dy + \frac{fT - g\Theta}{f - g}\, dz = 0,$$

viz. this is

$$[- tT(- tdz - zdt + P_1dt) - \theta\Theta(- \theta dz - zd\theta + \Pi'dt)](f - g) + (fT - g\Theta)\, dz = 0,$$

or collecting,

$$[(f + (f - g)t^2)\, T - (g + (g - f))\, \Theta]\, dz + (tTzdt + \theta\Theta zd\theta)(f - g)$$
$$- (tTP_1dt + \theta\Theta\Pi'd\theta)(f - g) = 0,$$

that is,

$$\left(\frac{1}{\Theta} - \frac{1}{T}\right) dz + (f - g)\, z\, (tTdt + \theta\Theta d\theta) - (f - g))(tTP_1dt + \theta\Theta\Pi'd\theta) = 0,$$

which is an integrable form as it should be, viz. the equation is

$$d\left(\frac{1}{T} - \frac{1}{\Theta}\right) z - (f - g)(tTP_1dt + \theta\Theta\Pi'd\theta) = 0,$$

and we obtain

$$\left(\frac{1}{T} - \frac{1}{\Theta}\right) z - (f - g)\int (tTP_1dt + \theta\Theta\Pi'd\theta) = 0,$$

the constant of integration being considered as included in the integral. But it is proper to alter the form of the second term. Take F, Φ arbitrary functions of t, θ respectively; and writing F_1, Φ' for the derived functions, assume $P = \frac{gF_1}{T^3}$, $\Pi = \frac{f\Phi'}{\Theta^3}$; we have

$$\int (tTP_1dt + \theta\Theta\Pi'd\theta) = \int \left(gtT\left(\frac{F_1}{T^3}\right)_1 dt + f\theta\Theta\left(\frac{\Phi'}{\Theta^3}\right)' d\theta\right)$$
$$= - F + \frac{gtF_1}{T^3} - \Phi + \frac{f\theta\Phi'}{\Theta^3}.$$

In fact this will be true if only

$$\left(- F + \frac{gtF_1}{T^3}\right)_1 = gtT\left(\frac{F_1}{T^3}\right)_1, \quad \left(- \Phi + \frac{f\theta\Phi'}{\Theta^3}\right)' = f\theta\Theta\left(\frac{\Phi'}{\Theta^3}\right)',$$

which are equations of like form in t, θ respectively, and it will be sufficient to verify the first of them. Effecting the differentiation, the terms in F_{11} destroy each other, and there remain only terms containing the factor F_1, and throwing this out we obtain

$$- 1 + \frac{g}{T^3} + \frac{gtT_1}{T^3} = 0,$$

viz. this is

$$- 1 + g(f + (f - g)\, t^2) - gt^2\, (f - g) = 0,$$

which is identically true, and the equation is thus verified.

The foregoing result is

$$\left(\frac{1}{T}-\frac{1}{\Theta}\right)z+(f-g)\left\{F+\Phi-\frac{gtF_1}{T^3}-\frac{f\theta\Phi'}{\Theta^3}\right\}=0;$$

we then have

$$x+tz-\frac{gF_1}{T^3}=0,\quad y+\theta z-\frac{f\Phi'}{\Theta^3}=0,$$

and hence, repeating also the equation for z,

$$\begin{aligned}
&\left(\frac{1}{T}-\frac{1}{\Theta}\right)x+(f-g)\left\{-t(F+\Phi)\qquad\qquad+\frac{ft\theta\Phi'}{\Theta^3}\right\}+\left(-1+\frac{g}{\Theta T}\right)\frac{F_1}{T^3}=0,\\
&\left(\frac{1}{T}-\frac{1}{\Theta}\right)y+(f-g)\left\{-\theta(F+\Phi)+\frac{gt\theta F_1}{T^3}\qquad\qquad\right\}+\left(\quad 1-\frac{f}{\Theta T}\right)\frac{\Phi'}{\Theta^3}=0,\\
&\left(\frac{1}{T}-\frac{1}{\Theta}\right)z+(f-g)\left\{\qquad F+\Phi-\frac{gtF_1}{T^3}-\frac{f\theta\Phi'}{\Theta^3}\right\}\qquad\qquad\qquad=0,
\end{aligned}$$

equations which give the values of the coordinates x, y, z in terms of the parameters t, θ. It will be recollected that $fg=1$ (f or g being arbitrary), that the values of T, Θ are

$$\frac{1}{T^2}=f+(f-g)t^2,\quad \frac{1}{\Theta^2}=g+(g-f)\theta^2,$$

and that F, Φ denote arbitrary functions of t, θ respectively. I repeat also the foregoing equations

$$A,\ B,\ C=-tT\frac{f-g}{T-\Theta},\quad -\theta\Theta\frac{f-g}{T-\Theta},\quad \frac{fT-g\Theta}{T-\Theta}.$$

The equations may be presented under a different form; we have

$$\begin{aligned}
&-tTx-\theta\Theta y+\frac{fT-g\Theta}{f-g}z+F+\Phi=0,\\
&-fT^3(x+tz)+F_1=0,\\
&-g\Theta^3(y+\theta z)+\Phi'=0,
\end{aligned}$$

where it will be observed that the second and third equations are the derivatives of the first equation in regard to t and θ respectively. We thus have the required surface as the envelope of the plane represented by the first equation, regarding therein t, θ as variable parameters. Moreover, the second equation (which contains only the parameter t) represents the planes of the curves of curvature of the one set; and the third equation (which contains only the parameter θ) represents the planes of the curves of curvature of the other set. It is to be observed that from the equations for l, λ (viz. $A + tC = gt$ and $B + \theta C = f\theta$) it appears that for any plane of the first set the inclination to a tangent plane of the surface is $= \cos^{-1} \frac{gt}{\sqrt{1+t^2}}$, and that for any plane of the second set the inclination is $= \cos^{-1} \frac{f\theta}{\sqrt{1+\theta^2}}$.

It may be remarked that the last mentioned results may be arrived at by the consideration of an equation $Ax + By + Cz + D = 0$, where the coefficients are functions of t and θ (A a function of t only, and B a function of θ only) such that the derived equations $A_1x + C_1z + D_1 = 0$ and $B'x + C'z + D' = 0$ depend the former of them upon t only, and the latter of them upon θ only.

A very simple case of the equation is when $f = g = 1$; here $T = \Theta = 1$, and the surface is the envelope of the plane $z - tx - \theta y + F + \Phi = 0$.

Returning to the general form

$$-tTx - \theta\Theta y + \frac{fT - g\Theta}{f - g} z + F + \Phi = 0,$$

I transform this by introducing therein in place of t, θ two variable parameters α, β which are such that $k\alpha = -tT$, $k\beta = \theta\Theta$ $\left(k \text{ a constant which is presently put} = \frac{1}{\sqrt{f-g}}\right)$, we find

$$t^2 = \frac{fk^2\alpha^2}{1 - (f-g)k^2\alpha^2}, \quad \theta^2 = \frac{gk^2\beta^2}{1 - (g-f)k^2\beta^2},$$

and thence

$$T = \frac{1}{\sqrt{f}} \sqrt{1 - (f-g)k^2\alpha^2}, \quad \Theta = \frac{1}{\sqrt{g}} \sqrt{1 - (g-f)k^2\beta^2},$$

or putting $k = \frac{1}{\sqrt{f-g}}$, these last values are

$$T = \frac{1}{\sqrt{f}} \sqrt{1 - \alpha^2}, \quad \Theta = \frac{1}{\sqrt{g}} \sqrt{1 + \beta^2},$$

and we hence obtain

$$\begin{aligned} \frac{fT - g\Theta}{f - g} &= \frac{\sqrt{f}}{f-g} \sqrt{1-\alpha^2} - \frac{\sqrt{g}}{f-g} \sqrt{1+\beta^2}, \\ &= \frac{1}{\sqrt{f-g}} \left\{ \frac{\sqrt{f}}{\sqrt{f-g}} \sqrt{1-\alpha^2} - \frac{\sqrt{g}}{\sqrt{f-g}} \sqrt{1+\beta^2} \right\}, \end{aligned}$$

say this is $\quad = k\{\lambda\sqrt{1-\alpha^2} - \mu\sqrt{1+\beta^2}\}$,

where $\lambda = \dfrac{\sqrt{f}}{\sqrt{f-g}}$, $\mu = \dfrac{\sqrt{g}}{\sqrt{f-g}}$, and therefore $\lambda^2 - \mu^2 = 1$ or $\mu = \sqrt{\lambda^2 - 1}$.

Hence writing $F + \Phi = k\,(A + B)$, k times the sum of two arbitrary functions of α and β respectively, the equation becomes

$$\alpha x - \beta y + z\,\{\lambda\sqrt{1-\alpha^2} - \sqrt{\lambda^2-1}\,\sqrt{1+\beta^2}\} + A + B = 0$$

viz. the surface is given as the envelope of this plane considering α, β as two variable parameters. This is the solution given by Darboux, "Leçons sur la théorie générale des surfaces, etc.," Paris, 1887, pp. 128–131. He obtains it in a very elegant manner, starting from the following theorem: Take A, A_1, etc., functions of the parameter α, and B, B_1, etc., functions of the parameter β; then if we have identically

$$(A_1 - B_1)^2 + (A_2 - B_2)^2 + (A_3 - B_3)^2 = (A_4 - B_4)^2,$$

the required surface will be obtained as the envelope of the plane

$$(A_1 - B_1)\,x + (A_2 - B_2)\,y + (A_3 - B_3)\,z = A - B,$$

where A, B are two new functions of α, β respectively.

The foregoing identity is the condition in order that each sphere of the one series $(x - A_1)^2 + (y - A_2)^2 + (z - A_3)^2 = A_4^2$ may touch each sphere of the other series $(x - B_1)^2 + (y - B_2)^2 + (z - B_3)^2 = B_4^2$; the two series of spheres thus envelope one and the same surface which will have its curves of curvature of each set circles: viz. this will be the surface of the fourth order called Dupin's Cyclide, the normals whereof pass through an ellipse and hyperbola which are focal curves one of the other, and which contain the centres of all the spheres touching the surface along its curves of curvature. The equations of the ellipse and hyperbola may be taken to be

$$\left(x^2 + \frac{z^2}{\lambda^2} = 1,\ y = 0\right) \text{ and } \left(y^2 - \frac{z^2}{\lambda^2 - 1} = -1,\ x = 0\right)$$

respectively, and we thence obtain the required PP surface as the envelope of the plane

$$\alpha x - \beta y + (\lambda\sqrt{1-\alpha^2} - \sqrt{\lambda^2 - 1}\sqrt{1+\beta^2})\,z + A + B = 0.$$

THE CASE $PP1^0$ = SERRET'S FIRST CASE OF PP.

We deduce this from the second case by writing therein $m = 0$, that is, $g = 0$, $f = \infty$; but it is necessary to make also a transformation upon the parameter θ, viz. in place thereof we introduce the new parameter ϕ, where $\theta^2 = \frac{g\phi^2}{f - g\phi^2}$, this gives

$$\frac{1}{\Theta^2} = g + (g - f)\theta^2 = g\left(1 + \frac{(g-f)\phi^2}{f - g\phi^2}\right) = \frac{gf(1-\phi^2)}{f - g\phi^2}, \quad \theta^2 = \frac{g\phi^2}{f - g\phi^2}$$

and thence

$$\theta\Theta = \frac{\theta}{\sqrt{g + (g-f)\theta^2}} = \frac{\phi}{\sqrt{f}\sqrt{1-\phi^2}}; \quad \frac{fT - g\Theta}{f - g} \text{ for } g = 0 \text{ is } = T.$$

We have also $T = \frac{1}{\sqrt{f + (f-g)t^2}}, = \frac{1}{\sqrt{f}\sqrt{1+t^2}}$ when $g = 0$, and substituting these values, considering Φ as a function of ϕ, and for $F + \Phi$ writing as we may do $\frac{F + \Phi}{\sqrt{f}}$, the equation becomes

$$\frac{-t}{\sqrt{f}\sqrt{1+t^2}}x - \frac{\phi y}{\sqrt{f}\sqrt{1-\phi^2}} + \frac{z}{\sqrt{f}\sqrt{1+t^2}} + \frac{F + \Phi}{\sqrt{f}} = 0,$$

where the divisor $\sqrt{f}$ is to be omitted. Hence finally, instead of ϕ restoring the original letter θ, and again considering Φ as a function of θ, the equation is

$$\frac{z - tx}{\sqrt{1+t^2}} - \frac{\theta y}{\sqrt{1-\theta^2}} + F + \Phi = 0,$$

viz. here F, Φ are arbitrary functions of t, θ respectively, and the surface is the envelope of this plane considering t, θ as variable.

We obtain an imaginary special form of $PP1^0$ by writing in this equation $k\theta$ for θ and then putting $k = \infty$; the Φ remains an arbitrary function of the new θ, and the equation is

$$\frac{z - tx}{\sqrt{1+t^2}} + iy + F + \Phi = 0$$

($i = \sqrt{-1}$ as usual). This is in fact the equation which is obtained from $PP3^0$ by simply writing therein $g = 0$ without the transformation upon θ.

PS. THE SETS OF CURVES OF CURVATURE, THE FIRST PLANE, THE SECOND SPHERICAL.

The six equations are

$$\begin{aligned}
&A^2+B^2+C^2 &&=1,\\
&ax+by+cz+u &&=0,\\
&Aa+Bb+Cc+l &&=0,\\
&x^2+y^2+z^2-2\alpha x-2\beta y-2\gamma z-2v &&=0,\\
&A(x-\alpha)+B(y-\beta)+C(z-\gamma)-\lambda &&=0,\\
&Adx+Bdy+Cdz &&=0.
\end{aligned}$$

The condition is

$$a\alpha+b\beta+c\gamma-l\lambda+u=0,$$

(not containing v so that this remains an arbitrary function of θ). The cases are

	a	b	c	l	u	α	β	γ	λ
$PS1^0$	a	b	c	0	0	0	0	0	λ
$PS2^0$	a	b	c	l	ml	0	0	0	m
$PS3^0$	a	b	c	$-mc$	0	0	0	γ	$\frac{1}{m}\gamma$
$PS4^0$	a	b	0	l	ml	0	0	γ	m
$PS5^0$	a	b	0	0	0	0	0	γ	λ
$PS6^0$	0	b	0	l	ml	α	0	γ	m
$PS7^0$	a	b	0	ma	0	α	0	γ	$-\frac{1}{m}\alpha$

where m is an arbitrary constant and in the body of the table the other italic letters are arbitrary functions of t, and the greek letters arbitrary functions of θ.

$PS1^0$ is Serret's first case of *PS*, included in his second case.
$PS2^0$ gives developable.
$PS3^0$ is Serret's second case of *PS*.
$PS4^0$ is Serret's third case of *PS*.
$PS5^0$ gives circular sections (surfaces of resolution).
$PS6^0$ gives circular sections (tubular surfaces).
$PS7^0$ gives circular sections.

I consider

$$PS3^0 = \text{SERRET'S SECOND CASE OF } PS.$$

The six equations are

$$\begin{aligned} A^2 + B^2 + C^2 &= 1, \\ ax + by + cz &= 0, \\ Aa + Bb + Cc &= -cm, \\ x^2 + y^2 + (z - m\phi)^2 &= \theta + m^2\phi^2, \\ Ax + By + C(z - m\phi) &= \phi, \\ Adx + Bdy + Cdz &= 0, \end{aligned}$$

(where a, b, c are assumed such that $a^2 + b^2 + c^2 = 1$). We easily obtain

$$\begin{aligned} (1 - c^2)A &= -ac(C + m) - b\sqrt{\Omega}, \\ (1 - c^2)B &= -bc(C + m) + a\sqrt{\Omega}, \end{aligned}$$

and thence

$$aB - bA = \sqrt{\Omega},$$

where

$$\Omega = (1 - c^2)(1 - C^2) - c^2(C + m)^2, \quad = 1 - c^2 - C^2 - 2c^2Cm - c^2m^2;$$

also

$$\begin{aligned} x\sqrt{1 - c^2m^2} &= A\phi\sqrt{1 - c^2m^2} + (bC - cB)\sqrt{\theta + (m^2 - 1)\phi^2}, \\ y\sqrt{1 - c^2m^2} &= B\phi\sqrt{1 - c^2m^2} + (cA - aC)\sqrt{\theta + (m^2 - 1)\phi^2}, \\ z\sqrt{1 - c^2m^2} &= (C + m)\phi\sqrt{1 - c^2m^2} + (aB - bA)\sqrt{\theta + (m^2 - 1)\phi^2}. \end{aligned}$$

We seek for the differential equation in C, t, θ. From the equation

$$Ax + By + (C - m\phi)z = \phi,$$

and attending to $Adx + Bdy + Cdz = 0$, we deduce

$$xdA + ydB + (z - m\phi)dC - (1 + Cm)\phi' d\theta = 0,$$

and we have herein to substitute for dA, dB their values in terms of dC, dt, $d\theta$. We have

$$\begin{aligned} AdA + BdB &= -CdC, \\ adA + bdB &= -cdC - Q, \end{aligned}$$

if for shortness $Q = Ada + Bdb + (C + m)dc$. Hence

$$\begin{aligned} \sqrt{\Omega}\,dA &= (-cB + bC)dC - BQ, \\ \sqrt{\Omega}\,dB &= (-aC + cA)dC + AQ. \end{aligned}$$

We find without difficulty,

$$(1 - c^2)Q = (C + m)dc + (adb - bda)\sqrt{\Omega},$$

and consequently,

$$\begin{aligned}(1-c^2)\sqrt{\Omega}\,dA &= \{\quad b(C+c^2m)-ac\sqrt{\Omega}\}\,dC\\ &\qquad\qquad -B\{(C+m)\,dc+(adb-bda)\sqrt{\Omega}\},\\ (1-c^2)\sqrt{\Omega}\,dB &= \{-a(C+c^2m)-bc\sqrt{\Omega}\}\,dC\\ &\qquad\qquad +A\{(C+m)\,dc+(adb-bda)\sqrt{\Omega}\}.\end{aligned}$$

Substituting these values we have

$$\begin{aligned}&\{(bx-ay)(C+c^2m)-(ax+by)\,c\sqrt{\Omega}\}\,dC\\ &\quad-(Bx-Ay)\{(C+m)\,dc+(adb-bda)\sqrt{\Omega}\}\\ &\quad+(1-c^2)\sqrt{\Omega}\{(z-m\phi)\,dC-(1+Cm)\,\phi'd\theta\}=0,\end{aligned}$$

viz. this is

$$\begin{aligned}&\{(bx-ay)(C+c^2m)-(ax+by)\,c\sqrt{\Omega}+(1-c^2)(z-m\phi)\sqrt{\Omega}\}\,dC\\ &\qquad\qquad-(Bx-Ay)\{(C+m)\,dc+(adb-bda)\sqrt{\Omega}\}\\ &\qquad\qquad-(1-c^2)(1+Cm)\sqrt{\Omega}\phi'd\theta=0.\end{aligned}$$

The coefficient of dC contains a term $-(ax+by+cz)\,c\sqrt{\Omega}$ which is $=0$. Moreover, we have

$$bx-ay=-\phi\sqrt{\Omega}+\frac{C+c^2m}{\sqrt{1-c^2m^2}}\sqrt{\theta+(m^2-1)\,\phi^2},$$

and then

$$\begin{aligned}(1-c^2)(Bx-Ay)&=-c\,(C+m)(bx-ay)-cz\sqrt{\Omega}\\ &=-c\,(C+m)\left\{-\phi\sqrt{\Omega}+\frac{C+c^2m}{\sqrt{1-c^2m^2}}\sqrt{\theta+(m^2-1)\,\phi^2}\right\}\\ &\quad-c\sqrt{\Omega}\left\{(C+m)\,\phi+\frac{\sqrt{\Omega}\sqrt{\theta+(m^2-1)\,\phi^2}}{\sqrt{1-c^2m^2}}\right\},\end{aligned}$$

which, observing that the terms in $C\phi\sqrt{\Omega}$ destroy each other, and that we have $(C+m)(C+c^2m)+\Omega=(1-c^2)(1+Cm)$, gives

$$Bx-Ay=\frac{-c\,(1+Cm)}{\sqrt{1-c^2m^2}}\sqrt{\theta+(m^2-1)\,\phi^2},$$

and the equation becomes

$$\begin{aligned}&\left\{\left(-\phi\sqrt{\Omega}+\frac{C+c^2m}{\sqrt{1-c^2m^2}}\sqrt{\theta+(m^2-1)\,\phi^2}\right)(C+c^2m)+z\sqrt{\Omega}\right.\\ &\quad\left.-(1-c^2)\,m\phi\sqrt{\Omega}\right\}dC\\ &\quad-c\,(1+Cm)\,\frac{\sqrt{\theta+(m^2-1)\,\phi^2}}{\sqrt{1-c^2m^2}}\left\{(C+m)\,dc+(adb-bda)\sqrt{\Omega}\right\}\\ &\quad-(1-c^2)\sqrt{\Omega}\,(1+Cm)\,\phi'd\theta=0.\end{aligned}$$

Here the coefficient of dC is $= [z-(C+m)\phi]\sqrt{\Omega}+\dfrac{(C+c^2m)^2}{\sqrt{1-c^2m^2}}\sqrt{\theta+(m^2-1)\phi^2}$, or substituting for $z-(C+m)\sqrt{\Omega}$ its value $=\dfrac{\sqrt{\Omega}\sqrt{\theta+(m^2-1)\phi^2}}{\sqrt{1-c^2m^2}}$, and observing that $\Omega+(C+c^2m)^2=(1-c^2m^2)(1-c^2)$, this coefficient is found to be $=\sqrt{1-c^2m^2}\,(1-c^2)\sqrt{\theta+(m^2-1)\phi^2}$, and we have

$$\begin{aligned}\sqrt{1-c^2m^2}\,(1-c^2)&\sqrt{\theta+(m^2-1)\phi^2}\,dC\\ &-c\,(1+Cm)\frac{\sqrt{\theta+(m^2-1)\phi^2}}{\sqrt{1-c^2m^2}}\Big\{(C+m)\,dc+(adb-bda)\sqrt{\Omega}\Big\}\\ &-(1-c^2)(1+Cm)\sqrt{\Omega}\phi' d\theta=0,\end{aligned}$$

or as this may be written

$$\begin{aligned}\frac{1}{\sqrt{\Omega}}\left\{\frac{\sqrt{1-c^2m^2}dC}{1+Cm}-\frac{(C+m)\,cdc}{(1-c^2)\sqrt{1-c^2m^2}}\right\}&-\frac{c\,(adb-bda)}{(1-c^2)\sqrt{1-c^2m^2}}\\ &-\frac{\phi' d\theta}{\sqrt{\theta+(m^2-1)\phi^2}}=0,\end{aligned}$$

where from the foregoing value of Ω we have identically

$$\Omega\,(1-m^2)=(1-c^2)(1+Cm)^2-(1-c^2m^2)(C+m)^2.$$

Here a, b, c are functions of t and we have thus the required differential equation in C, t, θ.

It is convenient to multiply by the constant factor $\sqrt{1-m^2}$. The first term is an exact differential, viz. writing

$$\sin\zeta=\frac{\sqrt{1-c^2m^2}}{\sqrt{1-c^2}}\,\frac{C+m}{1+Cm}\,,\ \text{ and therefore }\cos\zeta=\frac{\sqrt{1-m^2}\sqrt{\Omega}}{\sqrt{1-c^2}\,(1+Cm)}\,,$$

we have

$$d\zeta=\frac{\sqrt{1-m^2}}{\sqrt{\Omega}}\left\{\frac{\sqrt{1-c^2m^2}\,dC}{1+Cm}+\frac{(C+m)\,cdc}{(1-c^2)\sqrt{1-c^2m^2}}\right\},$$

as may easily be verified. And the second and third terms are obviously the differentials of a function of t and a function of θ respectively. But to obtain the integral functions, a transformation of each term is required.

First, for the term $\dfrac{\sqrt{1-m^2}\,c\,(adb-bda)}{(1-c^2)\sqrt{1-c^2m^2}}$; we take a, b, c functions of t which are such $\mathrm{a}^2+\mathrm{b}^2+\mathrm{c}^2=1$; and then writing a_1, b_1, c_1 for the derived functions so that $\mathrm{aa}_1+\mathrm{bb}_1+\mathrm{cc}_1=0$, we assume $\mathrm{a}', \mathrm{b}', \mathrm{c}'=V\mathrm{a}_1,\ V\mathrm{b}_1,\ V\mathrm{c}_1$ where $\dfrac{1}{V^2}=\mathrm{a}_1^2+\mathrm{b}_1^2+\mathrm{c}_1^2$; we have therefore $\mathrm{aa}'+\mathrm{bb}'+\mathrm{cc}'=0$, and $\mathrm{a}'^2+\mathrm{b}'^2+\mathrm{c}'^2=1$;

and then writing a'', b'', $c'' = bc' - b'c$, $ca' - c'a$, $ab' - a'b$ respectively, we have $aa'' + bb'' + cc'' = 0$, $a'a'' + b'b'' + c'c'' = 0$, $a''^2 + b''^2 + c''^2 = 1$; thus $a, b, c, a', b', c', a'', b'', c''$ are a set of rectangular coefficients. We then write

$$a, b, c = \frac{1}{\rho}(a' + mb''),\ \frac{1}{\rho}(b' - ma''),\ \frac{1}{\rho}c',$$

determining ρ so that $a^2 + b^2 + c^2 = 1$ as above, viz. we thus have

$$\rho^2 = (1 + cm)^2 + c'^2m^2.$$

Observe that we thus have $\rho^2(1 - c^2) = \rho^2 - c'^2$ and $\rho^2(1 - c^2m^2) = (1 + cm)^2$.

Writing now

$$T = \tan^{-1}\frac{c + m}{c''\sqrt{1 - m^2}},\ \text{and therefore}\ \sin T = \frac{c + m}{\sqrt{\rho^2 - c'^2}},\ \cos T = \frac{c''\sqrt{1 - m^2}}{\sqrt{\rho^2 - c'^2}},$$

we find that

$$dT = \frac{\sqrt{1 - m^2}\, c\,(adb - bda)}{(1 - c^2)\sqrt{1 - c^2m^2}}.$$

The verification is somewhat long, but it is very interesting. We have

$$dT = \frac{\sqrt{1 - m^2}\,\{c''dc - (c + m)\,dc''\}}{\rho^2 - c'^2},$$

or observing that $c'' = ab' - a'b$, $= V(ab_1 - a_1b)$, $dc = c_1dt$, this is

$$dT = \frac{\sqrt{1 - m^2}}{\rho^2 - c^2}\,\{V(ab_1 - a_1b)\,c_1 - (c + m)[V_1(ab_1 - a_1b) + V(ab_{11} - a_{11}b)]\,dt,$$

where we have

$$\frac{1}{V^2} = a_1^2 + b_1^2 + c_1^2,\ \text{and therefore}\ -\frac{V_1}{V^3} = a_1a_{11} + b_1b_{11} + c_1c_{11};$$

also from $aa_1 + bb_1 + cc_1 = 0$, we have $a_1^2 + b_1^2 + c_1^2 + aa_{11} + bb_{11} + cc_{11} = 0$, and we thence obtain

$$\begin{aligned} dT = \frac{\sqrt{1 - m^2}\,V^3dt}{\rho^2 - c'^2}\,\{&-(ab_1 - a_1b)\,c_1\,(aa_{11} + bb_{11} + cc_{11}) \\ &-(c + m)[-(a_1a_{11} + b_1b_{11} + c_1c_{11})(ab_1 - a_1b) + (a_1^2 + b_1^2 + c_1^2)(ab_{11} - ba_{11})]\}, \end{aligned}$$

the term in [] is found to be $= -c_1\{a_{11}(bc_1 - b_1c) + b_{11}(ca_1 - c_1a) + c_{11}(ab_1 - a_1b)\}$, hence c_1 appears as a factor of the whole expression, and reducing the part independent of m, we find

$$\begin{aligned} dT = \frac{\sqrt{1 - m^2}\,V^3c_1dt}{\rho^2 - c'^2}\,\{(a_1b_{11} - a_{11}b_1) + m\,[&a_{11}(bc_1 - b_1c) \\ &+ b_{11}(ca_1 - c_1a) + c_{11}(ab_1 - a_1b)]\}. \end{aligned}$$

Next calculating the value of $adb - bda$, we have

$$a = \frac{V}{\rho}\{a_1 + m(ca_1 - c_1a)\},\quad b = \frac{V}{\rho}\{b_1 - m(bc_1 - b_1c)\},$$

or as these may be written

$$a = \frac{V}{\rho}\{\mathrm{a}_1(1+\mathrm{c}m) - \mathrm{ac}_1 m\}, \quad b = \frac{V}{\rho}\{\mathrm{b}_1(1+\mathrm{c}m) - \mathrm{bc}_1 m\},$$

and we thence easily obtain

$$adb - bda = \frac{V^2 dt}{\rho^2}(1+\mathrm{c}m)\{\mathrm{a}_1\mathrm{b}_{11} - \mathrm{a}_{11}\mathrm{b}_1)$$
$$+ m\,[\mathrm{a}_{11}(\mathrm{bc}_1 - \mathrm{b}_1\mathrm{c}) + \mathrm{b}_{11}(\mathrm{ca}_1 - \mathrm{c}_1\mathrm{a}) + \mathrm{c}_{11}(\mathrm{ab}_1 - \mathrm{a}_1\mathrm{b})]\},$$

viz: the factor in { } has the same value as in the expression for dT, and we thus have

$$\frac{dT}{adb - bda} = \frac{\sqrt{1-m^2}\,V\mathrm{c}_1\rho^2}{(1+\mathrm{c}m)(\rho^2 - \mathrm{c}'^2)} = \frac{c\sqrt{1-m^2}}{\sqrt{1-c^2m^2}\,(1-c^2)},$$

that is,

$$dT = \frac{\sqrt{1-m^2}\,c\,(adb - bda)}{(1-c^2)\sqrt{1-c^2m^2}},$$

the required equation.

Secondly, for the term $\dfrac{\sqrt{1-m^2}\,\phi' d\theta}{\sqrt{\theta + (m^2-1)\phi^2}}$, we introduce Φ a function of θ, such that writing Φ' for the derived function we have

$$\phi = \frac{\Phi - 2\theta\Phi'}{\sqrt{1 - 4(1-m^2)\Phi\Phi' + 4(1-m^2)\theta\Phi'^2}}, \quad = \frac{\Phi - 2\theta\Phi'}{\sqrt{M}} \text{ suppose,}$$

whence also

$$\sqrt{\theta + (m^2-1)\phi^2} = \frac{\sqrt{\theta + (m^2-1)\Phi^2}}{\sqrt{M}}, \quad \frac{\phi}{\sqrt{\theta + (m^2-1)\phi^2}} = \frac{\Phi - 2\theta\Phi'}{\sqrt{\theta + (m^2-1)\Phi^2}}.$$

Then writing

$$\sin\Theta = \frac{\Phi\sqrt{1-m^2}}{\sqrt{\theta}}, \quad \cos\Theta = \frac{\sqrt{\theta + (m^2-1)\Phi^2}}{\sqrt{\theta}},$$

$$\sin\Theta_0 = \frac{\phi\sqrt{1-m^2}}{\sqrt{\theta}}, \quad \cos\Theta_0 = \frac{\sqrt{\theta + (m^2-1)\phi^2}}{\sqrt{\theta}},$$

we find

$$\cos\Theta\, d\Theta = -\frac{\frac{1}{2}\sqrt{1-m^2}\,(\Phi - 2\theta\Phi')\,d\theta}{\theta\sqrt{\theta}},$$

that is,

$$d\Theta = \frac{-\frac{1}{2}\sqrt{1-m^2}\,(\Phi - 2\theta\Phi')\,d\theta}{\theta\sqrt{\theta + (m^2-1)\Phi^2}},$$

and similarly

$$\cos\Theta_0\, d\Theta_0 = \frac{-\frac{1}{2}\sqrt{1-m^2}\,(\phi - 2\theta\phi')\,d\theta}{\theta\sqrt{\theta}},$$

that is,

$$d\Theta_0 = \frac{-\frac{1}{2}\sqrt{1-m^2}\,(\phi - 2\theta\phi')\,d\theta}{\theta\sqrt{\theta + (m^2-1)\phi^2}}.$$

Hence

$$-d\Theta+d\Theta_0=\frac{\sqrt{1-m^2}}{2\theta}\left\{\frac{\Phi-2\theta\Phi'}{\sqrt{\theta+(m^2-1)\Phi^2}}-\frac{\phi-2\theta\phi'}{\sqrt{\theta+(m^2-1)\phi^2}}\right\}d\theta$$

$$=\frac{\sqrt{1-m^2}}{2\theta}\left\{\frac{\phi-(\phi-2\theta\phi')}{\sqrt{\theta+(m^2-1)\phi^2}}\right\}d\theta,\ =\frac{\sqrt{1-m^2}\,\phi'd\theta}{\sqrt{\theta+(m^2-1)\phi^2}},$$

the required equation.

We find, moreover,

$$\sin(\Theta-\Theta_0)=\frac{2\Phi'\sqrt{1-m^2}\sqrt{\theta+(m^2-1)\Phi^2}}{\sqrt{M}},\ \cos(\Theta-\Theta_0)=\frac{1-2(1-m^2)\Phi\Phi'}{\sqrt{M}},$$

which will be presently useful.

The differential equation now is $d\zeta-dT+d\Theta-d\Theta_0=0$, hence the integral equation (taking the constant of integration $=0$) is $\zeta=T-\Theta+\Theta_0$, or say

$$\sin\zeta=\sin(T-\Theta+\Theta_0),$$

viz. substituting for sin T and cos T their values, and observing that

$$\sin\zeta=\frac{\sqrt{1-c^2m^2}}{\sqrt{1-c^2}}\,\frac{C+m}{1+Cm},\ =\frac{1+cm}{\sqrt{\rho^2-c'^2}}\,\frac{C+m}{1+Cm},$$

the factor $\dfrac{1}{\sqrt{\rho^2-c'^2}}$ multiplies out, and we have

$$(1+cm)\frac{C+m}{1+Cm}=(c+m)\cos(\Theta-\Theta_0)-c'\sqrt{1-m^2}\sin(\Theta-\Theta_0).$$

And I further remark here that a former equation is

$$\Omega(1-m^2)=(1-c^2)(1+Cm)^2-(1-c^2m^2)(C+m)^2,$$

that is,

$$\Omega\frac{1-m^2}{(1+Cm)^2}=(1-c^2)\left\{1-\frac{(1-c^2m^2)(C+m)^2}{(1-c^2)(1+Cm)^2}\right\}=(1-c^2)\cos^2\zeta.$$

We thus have

$$\sqrt{\Omega}=\frac{1+Cm}{\sqrt{1-m^2}}\,\frac{\sqrt{\rho^2-c'^2}}{\rho}\cos\zeta,$$

$$=\frac{1+Cm}{\rho\sqrt{1-m^2}}\{c'\sqrt{1-m^2}\cos(\Omega-\Omega_0)+(c+m)\sin(\Omega-\Omega_0)\}.$$

We have thus C, and consequently also A, B, x, y, z all of them given as functions of t, θ; but the formulae admit of further development.

Write $$\text{ȣ}=\frac{C+m}{1+Cm},\text{ whence also }C=\frac{\text{ȣ}-m}{1-m\text{ȣ}}.$$

We have $C(1-ms)+m=s$, and hence $(1+cm)\{C(1-ms)+m\}, =(1+cm)s$, $=(c+m)\cos(\Theta-\Theta_0)-c''\sqrt{1-m^2}\sin(\Theta-\Theta_0)$. Using the value of C given by this equation, and calculating from it those of A, B; then writing for shortness

$$\begin{aligned}X &= a\sqrt{1-m^2}\cos(\Theta-\Theta_0)-(a''-mb')\ \sin(\Theta-\Theta_0),\\ Y &= b\sqrt{1-m^2}\cos(\Theta-\Theta_0)-(b''+ma')\ \sin(\Theta-\Theta_0),\\ Z &= \quad (c+m)\cos(\Theta-\Theta_0)-c''\sqrt{1-m^2}\sin(\Theta-\Theta_0),\end{aligned}$$

we have

$$\begin{aligned}A(1-ms)(1+cm) &= \sqrt{1-m^2}X,\\ B(1-ms)(1+cm) &= \sqrt{1-m^2}Y,\\ C(1-ms)(1+cm) &= \qquad Z-m(1+cm),\end{aligned}$$

to which I join

$$s(1+cm)= \qquad Z.$$

By way of verification observe that $A^2+B^2+C^2=1$, and that the equations give

$$(1-ms)^2(1+cm)^2=(1-m^2)(X^2+Y^2+Z^2)+m^2Z^2-2mZ(1+cm)+m^2(1+cm)^2;$$

we have $X^2+Y^2+Z^2=(1+cm)^2$, $Z=s(1+cm)$, and hence the identity $(1-ms)^2(1+cm)^2=(1-m^2+m^2s^2-2ms+m^2)(1+cm)^2$.

Proceeding to calculate the values of x, y, z, recollecting that $\sqrt{1-c^2m^2}$ $=\dfrac{1}{\rho}(1+cm)$, we have

$$\begin{aligned}x(1+cm) &= A\phi(1+cm)+\rho(bC-cB)\sqrt{\theta+(m^2-1)\phi^2},\\ &= A\phi(1+cm)+\{(b'-ma'')C-c'B\}\sqrt{\theta+(m^2-1)\phi^2},\end{aligned}$$

that is,

$$\begin{aligned}x(1+cm)(1-ms) &= \phi\sqrt{1-m^2}X+\frac{1}{1+cm}\{b'-ma'')(Z-m(1+cm))\\ &\qquad -c'\sqrt{1-m^2}Y\}\sqrt{\theta+(m^2-1)\phi^2}\\ &= \phi\sqrt{1-m^2}X+\frac{1}{1+cm}\{(b'-ma'')Z-c'\sqrt{1-m^2}Y\}\sqrt{\theta+(m^2-1)\phi^2}\\ &\qquad -m(b'-ma'')\sqrt{\theta+(m^2-1)\phi^2},\end{aligned}$$

where the term $(b'-ma'')Z-c'\sqrt{1-m^2}Y$ contains the factor $1+cm$; in fact this is

$$\begin{aligned}&=(b'-ma'')\{\quad (c+m)\cos(\Theta-\Theta_0)-c''\sqrt{1-m^2}\sin(\Theta-\Theta_0)\}\\ &\quad -c'\sqrt{1-m^2}\{b\sqrt{1-m^2}\cos(\Theta-\Theta_0)-(b''+ma')\ \sin(\Theta-\Theta_0)\}.\end{aligned}$$

The coefficient of the cosine is $(b'-ma'')(c+m)-bc'(1-m^2)$, which is

$$\begin{aligned}&= b'c-bc'+m(b'-ca'')+m^2(-a''+bc'), =-a''+m(b'-ca'')+m^2(-b'c),\\ &\qquad =(1+cm)(-a''+mb'),\end{aligned}$$

and similarly the coefficient of $\sqrt{1-m^2}$ sine is $-\mathrm{c}''(\mathrm{b}'-m\mathrm{a}'')+\mathrm{c}'(\mathrm{b}''+m\mathrm{a}')$, $=-\mathrm{b}'\mathrm{c}''+\mathrm{b}''\mathrm{c}'+m(\mathrm{a}'\mathrm{c}'+\mathrm{a}''\mathrm{c}'')$, $=-\mathrm{a}+m(-\mathrm{ac})$, $=(1+\mathrm{c}m)(-\mathrm{a})$. Calculating in like manner the values of y and z, and putting for shortness

$$\begin{aligned}
X_1 &= (-\mathrm{a}''+m\mathrm{b}') \quad \cos(\Theta-\Theta_0) - \mathrm{a}\sqrt{1-m^2}\sin(\Theta-\Theta_0),\\
Y_1 &= (-\mathrm{b}''-m\mathrm{a}') \quad \cos(\Theta-\Theta_0) - \mathrm{b}\sqrt{1-m^2}\sin(\Theta-\Theta_0),\\
Z &= (-\mathrm{c}''\sqrt{1-m^2})\cos(\Theta-\Theta_0) + \quad (\mathrm{c}+m)\sin(\Theta-\Theta_0),
\end{aligned}$$

we have

$$\begin{aligned}
x &= \phi\sqrt{1-m^2}X + X_1\sqrt{\theta+(m^2-1)\phi^2} - m(\mathrm{b}'-m\mathrm{a}'')\sqrt{\theta+(m^2-1)\phi^2},\\
y &= \phi\sqrt{1-m^2}Y + Y_1\sqrt{\theta+(m^2-1)\phi^2} + m(\mathrm{a}'+m\mathrm{b}'')\sqrt{\theta+(m^2-1)\phi^2},\\
z &= \sqrt{1-m^2}\{\phi\sqrt{1-m^2}Z + Z_1\sqrt{\theta+(m^2-1)\phi^2}\},
\end{aligned}$$

which are the required expressions of x, y, z in terms of t and θ. It will be noticed that X, X_1, Y, Y_1, Z, Z_1, each contain a term with $\cos(\Theta-\Theta_0)$ and one with $\sin(\Theta-\Theta_0)$, but as the terms in X_1, Y_1, Z_1 are each multiplied by $\sqrt{\theta+(m^2-1)\phi^2}$, the cosine and sine terms of X, X_1, of Y, Y_1 and of Z, Z_1 do not in any case unite into a single term.

I remark that we have identically

$$\begin{aligned}
aX + bY + c\sqrt{1-m^2}Z &= 0,\\
aX_1 + bY_1 + c\sqrt{1-m^2}Z_1 &= 0.
\end{aligned}$$

The foregoing values of x, y, z thus satisfy $ax+by+cz=0$, which is one of the six equations. The others of them might be verified without difficulty. I recall that we have $a, b, c = \frac{1}{\rho}(\mathrm{a}'+m\mathrm{b}''), \frac{1}{\rho}(\mathrm{b}'-m\mathrm{a}''), \frac{1}{\rho}\mathrm{c}'$; the six equations might therefore be written

$$\begin{aligned}
&A^2+B^2+C^2=1,\\
&(\mathrm{a}'+m\mathrm{b}'')x+(\mathrm{b}'-m\mathrm{a}'')y+\mathrm{c}'z=0,\\
&(\mathrm{a}'+m\mathrm{b}'')A+(\mathrm{b}'-m\mathrm{a}'')B+\mathrm{c}'C=-\mathrm{c}'m,\\
&x^2+y^2+(z-m\phi)^2 = \theta+m^2\phi^2,\\
&Ax+By+C(z-m\phi) = \phi,\\
&Adx+Bdy+Cdz = 0.
\end{aligned}$$

THE CASE $PS1^0$ = SERRET'S FIRST CASE OF PS.

This is at once deduced from $PS3^0$ by writing therein $m=0$; the formulae are a good deal more simple. We introduce as before the rectangular coefficients

$\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{a}', \mathrm{b}', \mathrm{c}', \mathrm{a}'', \mathrm{b}'', \mathrm{c}''$, and the values of a, b, c then are $\mathrm{a}', \mathrm{b}', \mathrm{c}'$. The six equations, using therein these values for a, b, c, are

$$\begin{aligned} A^2 + B^2 &+ C^2 = 1, \\ \mathrm{a}'x + \mathrm{b}'y &+ \mathrm{c}'z = 0, \\ \mathrm{a}'A + \mathrm{b}'B &+ \mathrm{c}'C = 0, \\ x^2 + y^2 &+ z^2 = \theta, \\ Ax + By &+ Cz = \phi. \\ Adx + Bdy &+ Cdz = 0. \end{aligned}$$

The function Φ is such that $\phi = \dfrac{\Phi - 2\theta\Phi'}{\sqrt{1 - 4\Phi\Phi' + 4\theta\Phi'^2}} = \dfrac{\Phi - 2\theta\Phi'}{\sqrt{M}}$. We have

$$\sin\Theta = \frac{\Phi}{\sqrt{\theta}} \quad \cos\Theta = \frac{\sqrt{\theta - \Phi^2}}{\sqrt{\theta}},$$

$$\sin\Theta_0 = \frac{\phi}{\sqrt{\theta}} \quad \cos\Theta_0 = \frac{\sqrt{\theta - \phi^2}}{\sqrt{\theta}},$$

and thence

$$\sin(\Theta - \Theta_0) = \frac{2\Phi'\sqrt{\theta - \Phi^2}}{\sqrt{M}}; \quad \cos(\Theta - \Theta_0) = \frac{1 - 2\Phi\Phi'}{\sqrt{M}}.$$

Also

$$\sin\zeta = \frac{C}{\sqrt{1 - \mathrm{c}^2}}, \quad \cos\zeta = \frac{\sqrt{1 - \mathrm{c}^2 - C^2}}{\sqrt{1 - \mathrm{c}^2}}; \quad \sin T = \frac{\mathrm{c}}{\sqrt{1 - \mathrm{c}'^2}}, \quad \cos T = \frac{\mathrm{c}''}{\sqrt{1 - \mathrm{c}^2}},$$

$$\zeta = T - \Theta + \Theta_0, \quad C = \mathrm{c}\cos(\Theta - \Theta_0) - \mathrm{c}''\sin(\Theta - \Theta_0),$$

$$\sqrt{1 - \mathrm{c}^2 - C^2} = \mathrm{c}''\cos(\Theta - \Theta_0) + \mathrm{c}\sin(\Theta - \Theta_0).$$

We have

$$\begin{aligned} A = X &= \mathrm{a}\cos(\Theta - \Theta_0) - \mathrm{a}''\sin(\Theta - \Theta_0); \quad X_1 = \mathrm{a}''\cos(\Theta - \Theta_0) + \mathrm{a}\sin(\Theta - \Theta_0), \\ B = Y &= \mathrm{b}\cos(\Theta - \Theta_0) - \mathrm{b}''\sin(\Theta - \Theta_0); \quad Y_1 = \mathrm{b}''\cos(\Theta - \Theta_0) + \mathrm{b}\sin(\Theta - \Theta_0), \\ C = Y &= \mathrm{c}\cos(\Theta - \Theta_0) - \mathrm{c}''\sin(\Theta - \Theta_0); \quad Z_1 = \mathrm{c}''\cos(\Theta - \Theta_0) + \mathrm{c}\sin(\Theta - \Theta_0), \end{aligned}$$

and then

$$\begin{aligned} x &= X\phi + X_1\sqrt{\theta - \phi^2}, \\ y &= Y\phi + Y_1\sqrt{\theta - \phi^2}, \\ z &= Z\phi + Z_1\sqrt{\theta - \phi^2}, \end{aligned}$$

which are the expressions of the coordinates in terms of the parameters t and θ.

(*To be continued.*)

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Band IX.

Bendixson, L., Sur une extension à l'infini de la formule d'interpolation de Gauss.

Berger, A., Déduction de quelques formules analytiques d'un théorème élémentaire de la théorie des nombres.

Dobriner, H., Die Flächen constanter Krümmung mit einem System sphärischer Krümmungslinien dargestellt mit Hilfe von Theta-functionen zweier Variabeln.

Gyldén, H., Untersuchungen über die Convergenz der Reihen, welche zur Darstellung der Coordinaten der Planeten angewendet werden.

Hacks, J., Einige Sätze über Summen von Divisoren.

Lindstedt, A., Über ein Theorem des Herrn Tisserand aus der Störungstheorie.

Loria, G., Sur une démonstration du théorème fondamental de la théorie des équations algébriques.

Markoff, A., Sur une question de maximum et de minimum proposée par M. Tchebycheff.

Mellin, Hj., Über einen Zusammenhang zwischen gewissen linearen Differential- und Differenzengleichungen.

Netto, E., Über orthogonale Substitutionen.

Poincaré, H., Sur les résidus des intégrales doubles.

Stieltjes, T. J., Note sur un développement de l'intégrale $\int_0^a e^{x^2}dx$.

Stieltjes, T. J., Sur les racines de l'équation $X_n = 0$.

Tchebycheff, P., Sur la représentation des valeurs limites des intégrales par des résidus intégraux. Traduit du russe par Sophie Kowalevski.

Tchebycheff, P., Sur les sommes composées des coefficients des séries à termes positifs.

Weber, H., Theorie der Abel'schen Zahlkörper. IV. Über die Bildung Abel'scher Körper mit gegebener Gruppe.

Zeller, Ch., Kalender-Formeln.

Band X.

Bohlin, K., Über die Bedeutung des Princips der lebendigen Kraft für die Frage von der Stabilität dynamischer Systeme.

Dobriner, H., Die Minimalflächen mit einem System sphärischer Krümmungslinien.

Hacks, J., Über Summen von grössten Ganzen.

Humbert, G., Sur les intégrales algébriques des différentielles algébriques.

Kobb, G., Sur le mouvement d'un point matériel sur une surface de révolution.

Koenigs, G., Sur une classe de formes de différentielles et sur la théorie des systèmes d'éléments.

Lecornu, L., Sur les surfaces possédant les mêmes plans de symétrie que l'un des polyèdres réguliers.

Lerch, M., Un théorème de la théorie des séries.

Lipschitz, R., Zur Theorie der krummen Oberflächen.

Lipschitz, R., Beweis eines Satzes aus der Theorie der Substitutionen.

Pincherle, S., Sur certaines opérations fonctionnelles, représentées par des intégrales définies.

Poincaré, H., Remarques sur les intégrales irrégulières des équations linéaires.

Schwering, K., Über gewisse trinomische komplexe Zahlen.

Staude, O., Über eine Gattung transcendenter Raumcoordinaten.

Stenberg, E. A., Sur un cas spécial de l'équation différentielle de Lamé.

Stern, M. A., Sur la valeur de quelques séries qui dépendent de la fonction $E(x)$.

Stieltjes, T. J., Table des valeurs des sommes $S_s = \sum_{1} n^{-s}$.

Weingarten, J., Zur Theorie des Flächenpotentials.

Eneström, G., Inhaltsverzeichniss der Bände 1—10. —Table des matières dès tomes 1—10.

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CONTENTS.

PAGE

Memoir on a New Theory of Symmetric Functions. By CAPT. P. A. MACMAHON, R. A., ... 1

On the Integrals in Series of Binomial Differential Equations. By WILLIAM WOOLSEY JOHNSON, ... 37

Sur certaines courbes qu'on peut adjoindre aux courbes planes pour l'étude de leurs propriétés infinitésimales. Par M. MAURICE D'OCAGNE, ... 55

On the Surfaces with Plane or Spherical Curves of Curvature. By PROF. CAYLEY, 71

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Professore della R. Università di Roma

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JANUARY 1889.

I consider

$$PS3^0 = \text{SERRET'S SECOND CASE OF } PS.$$

The six equations are

$$\begin{aligned} A^2 + B^2 + C^2 &= 1, \\ ax + by + cz &= 0, \\ Aa + Bb + Cc &= -cm, \\ x^2 + y^2 + (z - m\phi)^2 &= \theta + m^2\phi^2, \\ Ax + By + C(z - m\phi) &= \phi, \\ Adx + Bdy + Cdz &= 0, \end{aligned}$$

(where a, b, c are assumed such that $a^2 + b^2 + c^2 = 1$). We easily obtain

$$\begin{aligned} (1 - c^2)A &= -ac(C + m) - b\sqrt{\Omega}, \\ (1 - c^2)B &= -bc(C + m) + a\sqrt{\Omega}, \end{aligned}$$

and thence

$$aB - bA = \sqrt{\Omega},$$

where

$$\Omega = (1 - c^2)(1 - C^2) - c^2(C + m)^2, \ = 1 - c^2 - C^2 - 2c^2Cm - c^2m^2;$$

also

$$\begin{aligned} x\sqrt{1 - c^2m^2} &= A\phi\sqrt{1 - c^2m^2} + (bC - cB)\sqrt{\theta + (m^2 - 1)\phi^2}, \\ y\sqrt{1 - c^2m^2} &= B\phi\sqrt{1 - c^2m^2} + (cA - aC)\sqrt{\theta + (m^2 - 1)\phi^2}, \\ z\sqrt{1 - c^2m^2} &= (C + m)\phi\sqrt{1 - c^2m^2} + (aB - bA)\sqrt{\theta + (m^2 - 1)\phi^2}. \end{aligned}$$

We seek for the differential equation in C, t, θ. From the equation

$$Ax + By + (C - m\phi)z = \phi,$$

and attending to $Adx + Bdy + Cdz = 0$, we deduce

$$xdA + ydB + (z - m\phi)dC - (1 + Cm)\phi' d\theta = 0,$$

and we have herein to substitute for dA, dB their values in terms of dC, dt, $d\theta$. We have

$$\begin{aligned} AdA + BdB &= -CdC, \\ adA + bdB &= -cdC - Q, \end{aligned}$$

if for shortness $Q = Ada + Bdb + (C + m)dc$. Hence

$$\begin{aligned} \sqrt{\Omega}\,dA &= (-cB + bC)dC - BQ, \\ \sqrt{\Omega}\,dB &= (-aC + cA)dC + AQ. \end{aligned}$$

We find without difficulty,

$$(1 - c^2)Q = (C + m)dc + (adb - bda)\sqrt{\Omega},$$

and consequently,

$$
\begin{aligned}
(1-c^2)\sqrt{\Omega}\,dA &= \{\quad b(C+c^2m) - ac\sqrt{\Omega}\}\,dC \\
&\qquad\qquad - B\{(C+m)\,dc + (adb-bda)\sqrt{\Omega}\}, \\
(1-c^2)\sqrt{\Omega}\,dB &= \{-a(C+c^2m) - bc\sqrt{\Omega}\}\,dC \\
&\qquad\qquad + A\{(C+m)\,dc + (adb-bda)\sqrt{\Omega}\}.
\end{aligned}
$$

Substituting these values we have

$$
\begin{aligned}
&\{(bx-ay)(C+c^2m) - (ax+by)\,c\sqrt{\Omega}\}\,dC \\
&\quad - (Bx-Ay)\{(C+m)\,dc + (adb-bda)\sqrt{\Omega}\} \\
&\quad + (1-c^2)\sqrt{\Omega}\{(z-m\phi)\,dC - (1+Cm)\,\phi' d\theta\} = 0,
\end{aligned}
$$

viz. this is

$$
\begin{aligned}
&\{(bx-ay)(C+c^2m) - (ax+by)\,c\sqrt{\Omega} + (1-c^2)(z-m\phi)\sqrt{\Omega}\}\,dC \\
&\qquad\qquad - (Bx-Ay)\{(C+m)\,dc + (adb-bda)\sqrt{\Omega}\} \\
&\qquad\qquad - (1-c^2)(1+Cm)\sqrt{\Omega}\phi' d\theta = 0.
\end{aligned}
$$

The coefficient of dC contains a term $-(ax+by+cz)\,c\sqrt{\Omega}$ which is $=0$. Moreover, we have

$$
bx - ay = -\phi\sqrt{\Omega} + \frac{C+c^2m}{\sqrt{1-c^2m^2}}\sqrt{\theta+(m^2-1)\phi^2},
$$

and then

$$
\begin{aligned}
(1-c^2)(Bx-Ay) &= -c(C+m)(bx-ay) - cz\sqrt{\Omega} \\
&= -c(C+m)\left\{-\phi\sqrt{\Omega} + \frac{C+c^2m}{\sqrt{1-c^2m^2}}\sqrt{\theta+(m^2-1)\phi^2}\right\} \\
&\qquad - c\sqrt{\Omega}\left\{(C+m)\,\phi + \frac{\sqrt{\Omega}\sqrt{\theta+(m^2-1)\phi^2}}{\sqrt{1-c^2m^2}}\right\},
\end{aligned}
$$

which, observing that the terms in $C\phi\sqrt{\Omega}$ destroy each other, and that we have $(C+m)(C+c^2m)+\Omega = (1-c^2)(1+Cm)$, gives

$$
Bx - Ay = \frac{-c(1+Cm)}{\sqrt{1-c^2m^2}}\sqrt{\theta+(m^2-1)\phi^2},
$$

and the equation becomes

$$
\begin{aligned}
&\left\{\left(-\phi\sqrt{\Omega} + \frac{C+c^2m}{\sqrt{1-c^2m^2}}\sqrt{\theta+(m^2-1)\phi^2}\right)(C+c^2m) + z\sqrt{\Omega}\right. \\
&\quad \left. - (1-c^2)\,m\phi\sqrt{\Omega}\right\}dC \\
&\quad - c(1+Cm)\,\frac{\sqrt{\theta+(m^2-1)\phi^2}}{\sqrt{1-c^2m^2}}\left\{(C+m)\,dc + (adb-bda)\sqrt{\Omega}\right\} \\
&\quad - (1-c^2)\sqrt{\Omega}\,(1+Cm)\,\phi' d\theta = 0.
\end{aligned}
$$

Here the coefficient of dC is $= [z-(C+m)\phi]\sqrt{\Omega} + \dfrac{(C+c^2m)^2}{\sqrt{1-c^2m^2}}\sqrt{\theta+(m^2-1)\phi^2}$, or substituting for $z-(C+m)\sqrt{\Omega}$ its value $= \dfrac{\sqrt{\Omega}\sqrt{\theta+(m^2-1)\phi^2}}{\sqrt{1-c^2m^2}}$, and observing that $\Omega+(C+c^2m)^2 = (1-c^2m^2)(1-c^2)$, this coefficient is found to be $= \sqrt{1-c^2m^2}\,(1-c^2)\sqrt{\theta+(m^2-1)\phi^2}$, and we have

$$\begin{aligned}&\sqrt{1-c^2m^2}\,(1-c^2)\sqrt{\theta+(m^2-1)\phi^2}\,dC\\ &\qquad -c\,(1+Cm)\frac{\sqrt{\theta+(m^2-1)\phi^2}}{\sqrt{1-c^2m^2}}\left\{(C+m)\,dc+(adb-bda)\sqrt{\Omega}\right\}\\ &\qquad -(1-c^2)(1+Cm)\sqrt{\Omega}\phi' d\theta = 0,\end{aligned}$$

or as this may be written

$$\frac{1}{\sqrt{\Omega}}\left\{\frac{\sqrt{1-c^2m^2}dC}{1+Cm} - \frac{(C+m)\,cdc}{(1-c^2)\sqrt{1-c^2m^2}}\right\} - \frac{c\,(adb-bda)}{(1-c^2)\sqrt{1-c^2m^2}}$$
$$-\frac{\phi' d\theta}{\sqrt{\theta+(m^2-1)\phi^2}} = 0,$$

where from the foregoing value of Ω we have identically

$$\Omega\,(1-m^2) = (1-c^2)(1+Cm)^2-(1-c^2m^2)(C+m)^2.$$

Here a, b, c are functions of t and we have thus the required differential equation in C, t, θ.

It is convenient to multiply by the constant factor $\sqrt{1-m^2}$. The first term is an exact differential, viz. writing

$$\sin\zeta = \frac{\sqrt{1-c^2m^2}}{\sqrt{1-c^2}}\,\frac{C+m}{1+Cm}\,,\ \text{and therefore}\ \cos\zeta = \frac{\sqrt{1-m^2}\sqrt{\Omega}}{\sqrt{1-c^2}\,(1+Cm)}\,,$$

we have

$$d\zeta = \frac{\sqrt{1-m^2}}{\sqrt{\Omega}}\left\{\frac{\sqrt{1-c^2m^2}\,dC}{1+Cm} + \frac{(C+m)\,cdc}{(1-c^2)\sqrt{1-c^2m^2}}\right\},$$

as may easily be verified. And the second and third terms are obviously the differentials of a function of t and a function of θ respectively. But to obtain the integral functions, a transformation of each term is required.

First, for the term $\dfrac{\sqrt{1-m^2}\,c\,(adb-bda)}{(1-c^2)\sqrt{1-c^2m^2}}$; we take $\mathrm{a}, \mathrm{b}, \mathrm{c}$ functions of t which are such $\mathrm{a}^2+\mathrm{b}^2+\mathrm{c}^2=1$; and then writing $\mathrm{a}_1, \mathrm{b}_1, \mathrm{c}_1$ for the derived functions so that $\mathrm{aa}_1+\mathrm{bb}_1+\mathrm{cc}_1=0$, we assume $\mathrm{a}', \mathrm{b}', \mathrm{c}' = V\mathrm{a}_1, V\mathrm{b}_1, V\mathrm{c}_1$ where $\dfrac{1}{V^2} = \mathrm{a}_1^2+\mathrm{b}_1^2+\mathrm{c}_1^2$; we have therefore $\mathrm{aa}'+\mathrm{bb}'+\mathrm{cc}'=0$, and $\mathrm{a}'^2+\mathrm{b}'^2+\mathrm{c}'^2=1$;

and then writing $a'', b'', c'' = bc' - b'c,\ ca' - c'a,\ ab' - a'b$ respectively, we have $aa'' + bb'' + cc'' = 0$, $a'a'' + b'b'' + c'c'' = 0$, $a''^2 + b''^2 + c''^2 = 1$; thus $a, b, c, a', b', c', a'', b'', c''$ are a set of rectangular coefficients. We then write

$$a,\ b,\ c = \frac{1}{\rho}(a' + mb''),\ \frac{1}{\rho}(b' - ma''),\ \frac{1}{\rho}c',$$

determining ρ so that $a^2 + b^2 + c^2 = 1$ as above, viz. we thus have

$$\rho^2 = (1 + cm)^2 + c'^2m^2.$$

Observe that we thus have $\rho^2(1 - c^2) = \rho^2 - c'^2$ and $\rho^2(1 - c^2m^2) = (1 + cm)^2$.

Writing now

$$T = \tan^{-1}\frac{c + m}{c''\sqrt{1 - m^2}},\ \text{and therefore}\ \sin T = \frac{c + m}{\sqrt{\rho^2 - c'^2}},\ \cos T = \frac{c''\sqrt{1 - m^2}}{\sqrt{\rho^2 - c'^2}},$$

we find that

$$dT = \frac{\sqrt{1 - m^2}\,c\,(adb - bda)}{(1 - c^2)\sqrt{1 - c^2m^2}}.$$

The verification is somewhat long, but it is very interesting. We have

$$dT = \frac{\sqrt{1 - m^2}\,\{c''dc - (c + m)\,dc''\}}{\rho^2 - c'^2},$$

or observing that $c'' = ab' - a'b,\ = V(ab_1 - a_1b)$, $dc = c_1dt$, this is

$$dT = \frac{\sqrt{1 - m^2}}{\rho^2 - c^2}\,\{V(ab_1 - a_1b)\,c_1 - (c + m)[V_1(ab_1 - a_1b) + V(ab_{11} - a_{11}b)]\,dt,$$

where we have

$$\frac{1}{V^2} = a_1^2 + b_1^2 + c_1^2,\ \text{and therefore}\ -\frac{V_1}{V^3} = a_1a_{11} + b_1b_{11} + c_1c_{11};$$

also from $aa_1 + bb_1 + cc_1 = 0$, we have $a_1^2 + b_1^2 + c_1^2 + aa_{11} + bb_{11} + cc_{11} = 0$, and we thence obtain

$$\begin{aligned} dT = \frac{\sqrt{1 - m^2}\,V^3dt}{\rho^2 - c'^2}\,\{&-(ab_1 - a_1b)\,c_1\,(aa_{11} + bb_{11} + cc_{11}) \\ &-(c + m)[-(a_1a_{11} + b_1b_{11} + c_1c_{11})(ab_1 - a_1b) + (a_1^2 + b_1^2 + c_1^2)(ab_{11} - ba_{11})]\}, \end{aligned}$$

the term in [] is found to be $= -c_1\{a_{11}(bc_1 - b_1c) + b_{11}(ca_1 - c_1a) + c_{11}(ab_1 - a_1b)\}$, hence c_1 appears as a factor of the whole expression, and reducing the part independent of m, we find

$$\begin{aligned} dT = \frac{\sqrt{1 - m^2}\,V^3c_1dt}{\rho^2 - c'^2}\,\{(a_1b_{11} - a_{11}b_1) + m\,[&a_{11}(bc_1 - b_1c) \\ &+ b_{11}(ca_1 - c_1a) + c_{11}(ab_1 - a_1b)]\}. \end{aligned}$$

Next calculating the value of $adb - bda$, we have

$$a = \frac{V}{\rho}\{a_1 + m(ca_1 - c_1a)\},\quad b = \frac{V}{\rho}\{b_1 - m(bc_1 - b_1c)\},$$

or as these may be written

$$a = \frac{V}{\rho}\{\mathrm{a}_1(1+\mathrm{c}m) - \mathrm{ac}_1 m\},\quad b = \frac{V}{\rho}\{\mathrm{b}_1(1+\mathrm{c}m) - \mathrm{bc}_1 m\},$$

and we thence easily obtain

$$\begin{aligned} adb - bda = \frac{V^2 dt}{\rho^2}(1+\mathrm{c}m)\{&\mathrm{a}_1\mathrm{b}_{11} - \mathrm{a}_{11}\mathrm{b}_1) \\ &+ m\,[\mathrm{a}_{11}(\mathrm{bc}_1 - \mathrm{b}_1\mathrm{c}) + \mathrm{b}_{11}(\mathrm{ca}_1 - \mathrm{c}_1\mathrm{a}) + \mathrm{c}_{11}(\mathrm{ab}_1 - \mathrm{a}_1\mathrm{b})]\}, \end{aligned}$$

viz: the factor in { } has the same value as in the expression for dT, and we thus have

$$\frac{dT}{adb - bda} = \frac{\sqrt{1-m^2}\,V\mathrm{c}_1\rho^2}{(1+\mathrm{c}m)(\rho^2 - \mathrm{c}'^2)} = \frac{c\sqrt{1-m^2}}{\sqrt{1-c^2m^2}\,(1-c^2)},$$

that is,

$$dT = \frac{\sqrt{1-m^2}\,c\,(adb - bda)}{(1-c^2)\sqrt{1-c^2m^2}},$$

the required equation.

Secondly, for the term $\dfrac{\sqrt{1-m^2}\,\phi' d\theta}{\sqrt{\theta + (m^2-1)\phi^2}}$, we introduce Φ a function of θ, such that writing Φ' for the derived function we have

$$\phi = \frac{\Phi - 2\theta\Phi'}{\sqrt{1 - 4(1-m^2)\Phi\Phi' + 4(1-m^2)\theta\Phi'^2}}, \quad = \frac{\Phi - 2\theta\Phi'}{\sqrt{M}} \text{ suppose,}$$

whence also

$$\sqrt{\theta + (m^2-1)\phi^2} = \frac{\sqrt{\theta + (m^2-1)\Phi^2}}{\sqrt{M}},\quad \frac{\phi}{\sqrt{\theta + (m^2-1)\phi^2}} = \frac{\Phi - 2\theta\Phi'}{\sqrt{\theta + (m^2-1)\Phi^2}}.$$

Then writing

$$\sin\Theta = \frac{\Phi\sqrt{1-m^2}}{\sqrt{\theta}},\quad \cos\Theta = \frac{\sqrt{\theta + (m^2-1)\Phi^2}}{\sqrt{\theta}},$$

$$\sin\Theta_0 = \frac{\phi\sqrt{1-m^2}}{\sqrt{\theta}},\quad \cos\Theta_0 = \frac{\sqrt{\theta + (m^2-1)\phi^2}}{\sqrt{\theta}},$$

we find

$$\cos\Theta\, d\Theta = -\frac{\tfrac{1}{2}\sqrt{1-m^2}\,(\Phi - 2\theta\Phi')\,d\theta}{\theta\sqrt{\theta}},$$

that is,

$$d\Theta = \frac{-\tfrac{1}{2}\sqrt{1-m^2}\,(\Phi - 2\theta\Phi')\,d\theta}{\theta\sqrt{\theta + (m^2-1)\Phi^2}},$$

and similarly

$$\cos\Theta_0\, d\Theta_0 = \frac{-\tfrac{1}{2}\sqrt{1-m^2}\,(\phi - 2\theta\phi')\,d\theta}{\theta\sqrt{\theta}},$$

that is,

$$d\Theta_0 = \frac{-\tfrac{1}{2}\sqrt{1-m^2}\,(\phi - 2\theta\phi')\,d\theta}{\theta\sqrt{\theta + (m^2-1)\phi^2}}.$$

Hence

$$-d\Theta + d\Theta_0 = \frac{\sqrt{1-m^2}}{2\theta}\left\{\frac{\Phi - 2\theta\Phi'}{\sqrt{\theta + (m^2-1)\Phi^2}} - \frac{\phi - 2\theta\phi'}{\sqrt{\theta + (m^2-1)\phi^2}}\right\} d\theta$$

$$= \frac{\sqrt{1-m^2}}{2\theta}\left\{\frac{\phi - (\phi - 2\theta\phi')}{\sqrt{\theta + (m^2-1)\phi^2}}\right\} d\theta, \quad = \frac{\sqrt{1-m^2}\,\phi' d\theta}{\sqrt{\theta + (m^2-1)\phi^2}},$$

the required equation.

We find, moreover,

$$\sin(\Theta - \Theta_0) = \frac{2\Phi'\sqrt{1-m^2}\sqrt{\theta + (m^2-1)\Phi^2}}{\sqrt{M}}, \quad \cos(\Theta - \Theta_0) = \frac{1 - 2(1-m^2)\Phi\Phi'}{\sqrt{M}},$$

which will be presently useful.

The differential equation now is $d\zeta - dT + d\Theta - d\Theta_0 = 0$, hence the integral equation (taking the constant of integration $= 0$) is $\zeta = T - \Theta + \Theta_0$, or say

$$\sin\zeta = \sin(T - \Theta + \Theta_0),$$

viz. substituting for $\sin T$ and $\cos T$ their values, and observing that

$$\sin\zeta = \frac{\sqrt{1-c^2m^2}}{\sqrt{1-c^2}}\,\frac{C+m}{1+Cm}, \quad = \frac{1+\mathrm{c}m}{\sqrt{\rho^2-\mathrm{c}'^2}}\,\frac{C+m}{1+Cm},$$

the factor $\frac{1}{\sqrt{\rho^2 - \mathrm{c}'^2}}$ multiplies out, and we have

$$(1+\mathrm{c}m)\frac{C+m}{1+Cm} = (\mathrm{c}+m)\cos(\Theta-\Theta_0) - \mathrm{c}''\sqrt{1-m^2}\sin(\Theta-\Theta_0).$$

And I further remark here that a former equation is

$$\Omega(1-m^2) = (1-\mathrm{c}^2)(1+Cm)^2 - (1-\mathrm{c}^2m^2)(C+m)^2,$$

that is,

$$\Omega\frac{1-m^2}{(1+Cm)^2} = (1-\mathrm{c}^2)\left\{1 - \frac{(1-\mathrm{c}^2m^2)(C+m)^2}{(1-\mathrm{c}^2)(1+Cm)^2}\right\} = (1-\mathrm{c}^2)\cos^2\zeta.$$

We thus have

$$\sqrt{\Omega} = \frac{1+Cm}{\sqrt{1-m^2}}\,\frac{\sqrt{\rho^2-\mathrm{c}'^2}}{\rho}\cos\zeta,$$

$$= \frac{1+Cm}{\rho\sqrt{1-m^2}}\{\mathrm{c}''\sqrt{1-m^2}\cos(\Omega-\Omega_0) + (\mathrm{c}+m)\sin(\Omega-\Omega_0)\}.$$

We have thus C, and consequently also A, B, x, y, z all of them given as functions of t, θ; but the formulae admit of further development.

Write $$s = \frac{C+m}{1+Cm}, \text{ whence also } C = \frac{s-m}{1-ms}.$$

éléments. Les deux suites conduiront donc au même résultat. Il en serait de même si l'on avait

$$a_h = a_l, \quad a_\iota = a_k.$$

2me cas

$$a_h = a_k$$

tandis que a_ι est différent de a_l. Dans ce cas je déduis de la suite (A') la suite (A'') suivante

$$a_{h,\iota,l}, \ldots$$

où le pointillé indique tous les éléments de la suite (A), moins a_h, a_ι et a_l. De même de la suite $(\mathfrak{A}')$ je déduis la suite $(\mathfrak{A}'')$ suivante

$$a_{k,l,\iota}, \ldots$$

où le pointillé indique tous les éléments de la suite (A) moins a_k (qui est égal à a_h), a_l et a_ι. Or on a

$$a_{h,\iota,l} = a_{k,l,\iota}$$

les suites (A'') et $(\mathfrak{A}'')$ ne peuvent donc différer que par l'ordre de leurs éléments et conduiront par conséquent au même résultat. La supposition

$$a_h = a_l$$

et a_ι différent de a_k rentre dans le même cas.

3me cas. L'élément a_h est différent de a_k et de a_l et l'élément a_ι est différent de a_k et de a_l. Dans ce cas je déduis de la suite (A') la suite (A'') suivante

$$a_{h,\iota}, \; a_{k,l}, \ldots$$

où le pointillé indique[*] tous les éléments de la suite (A), moins a_h, a_ι, a_k et a_l. De même de la suite $(\mathfrak{A}')$, je déduis la suite $(\mathfrak{A}'')$ suivante

$$a_{h,\iota}, \; a_{k,l}, \ldots$$

où le pointillé[†] indique en core tous les éléments de la suite (A) moins a_h, a_ι, a_k et a_l. Les suites (A'') et $(\mathfrak{A}'')$ ne peuvent donc différer que par l'ordre de leurs éléments et conduiront par conséquent au même résultat. La proposition se trouve ainsi démontrée dans toute sa généralite.[‡]

---

[*] Pour $v = 3$ le pointillé est à supprimer.

[†] Pour $v = 3$ le pointillé est à supprimer.

[‡] Cf. *Lejeune Dirichlet*, Vorlesungen über Zahlentheorie, herausgegeben von *Dedekind*, 3te Auflage, Braunschweig, 1879.

En particulier, on peut distribuer les éléments de la suite (A) en un certain nombre u de *compagnies**

$$B_1, B_2, \ldots . B_{u-1}, B_u$$

puis réunir les éléments de chaque compagnie B_k en un seul élément b_k; on aura toujours

$$a_{1,2,\ldots . m} = b_{1,2,\ldots . u}.$$

3.

La grande analogie qui existe entre la composition et la multiplication permet d'emprunter à la théorie de cette dernière beaucoup de termes déjà consacrés par l'usage. Si, en composant l'élément b_1 avec l'élément b_2, on obtient l'élément c, nous dirons qu'en multipliant b_1 avec b_2 on obtient c; b_1 sera le multiplicande, b_2 le multiplicateur et c le produit. Mais comme le produit c ne change pas, si l'on prend b_2 pour multiplicande et b_1 pour multiplicateur, on peut donner tant à b_1 qu'à b_2 le nom général de facteur. D'une manière générale si, en réunissant m éléments

$$b_1, b_2, \ldots . b_{m-1}, b_m$$

en un seul, on obtient c, l'élément bien déterminé c portera le nom de produit, et chacun des éléments b sera désigné sous le nom de facteur. Obtenir l'élément c à l'aide des éléments

$$b_1, b_2, \ldots . b_{m-1}, b_m$$

par la réunion de ces derniers en un seul, s'appelle effecteur le produit des dits éléments. Nous avons vu qu'il y a plusieurs manières de réunir les éléments

$$b_1, b_2, \ldots . b_{m-1}, b_m$$

en un seul, mais que toutes ces manières conduisent au même résultat. A chacune de ces manières correspond une certaine manière *d'effectuer le produit* des éléments

$$b_1, b_2, \ldots . b_{m-1}, b_m.$$

En écrivant

$$b_1 b_2 \ldots . b_{m-1} b_m$$

nous supposons qu'on effectuera le produit en composant b_1 avec b_2, puis le résultat de composition avec b_3, et ainsi de suite. Mais toutes les fois qu'il s'agit uniquement du produit, la manière de l'effectuer est indifférente. L'égalité

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* J'évite le mot groupe.

$$b_1 b_2 \ldots . b_{m-1} b_m = c$$

peut donc être considérée comme exprimant ce fait que, en réunissant les éléments $b_1, b_2, \ldots . b_{m-1}, b_m$ en un seul, on obtient l'élément c. Si, en composant b_1 avec b_1, puis le résultat de composition avec b_1, et ainsi de suite m fois, on obtient l'élément c, nous dirons qu'en élevant l'élément b_1 à la *puissance* m on obtient c et nous écrirons

$$b_1^m = c.$$

L'élément c portera le nom de puissance m-ème de b_1. On aura d'ailleurs, comme en algèbre,

$$b_1^h b_1^k = b_1^{h+k},$$

$$(b_1 b_2)^h = b_1^h b_2^h$$

et

$$(b_1^h)^k = b_1^{hk}.$$

Tout élément b du groupe Ω peut donc être donné soit explicitement soit implicitement sous forme d'un produit non effectué de plusieurs éléments qui à leur tous peuvent être donnés soit explicitement soit implicitement sous forme d'un produit de plusieurs éléments et ainsi de suite jusqu'à ce qu'on parvient à des éléments donnés explicitement. Si un élément b donné explicitement ou implicitement est identique* à un élément c donné soit explicitement soit implicitement pourvu qu'on exécute toutes les opérations indiquées, nous écrirons

$$b = c \qquad (\text{gr. } \Omega)$$

afin de mettre en évidence le groupe Ω. Une telle formule portera le nom d'égalité. Si l'on a

$$b = c \qquad (\text{gr. } \Omega)$$

et

$$b_1 = c_1 \qquad (\text{gr. } \Omega)$$

on peut en tirer

$$bb_1 = cc_1 \qquad (\text{gr. } \Omega).$$

En effet, si l'on obtient les éléments b et b_1 sous forme explicite et que l'on effectue le produit bb_1, le résultat sera identique à celui auquel on arrivera en obtenant d'abord c et c_1 sous forme explicite et en affectuant le produit cc_1. Il en résulte que l'égalité

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* Les identités que nous considérons seront toujours d'une telle nature que si un élément est identique à un autre, cet autre est identique au premier et que deux éléments identiques à un troisième, sont identiques l'un à l'autre.

$$bb_1 = cc_1 \qquad (\text{gr. } \Omega),$$

a lieu de quelque manière que l'on réunisse en un seul les éléments qui figurent dans b et b_1 d'un côté et aux qui figurent dans c et c_1 de l'autre côté. De même, si l'on a

$$\begin{aligned} b &= c \qquad (\text{gr. } \Omega), \\ b_1 &= c_1 \qquad (\text{gr. } \Omega), \\ &\ldots\ldots\ldots\ldots \\ b_{m-1} &= c_{m-1} \qquad (\text{gr. } \Omega), \end{aligned}$$

on en tirera

$$bb_1 \ldots . b_{m-1} = cc_1 \ldots . c_{m-1} \qquad (\text{gr. } \Omega)$$

et on peut réunir en un seul tant les éléments qui figurent dans $b, b_1, \ldots . b_{m-1}$ que ceux qui figurent dans $c, c_1 \ldots . c_{m-1}$ d'une manière quelconque. Cette proposition s'énonce en disant qu'on peut multiplier un nombre quelconque d'égalités membre à membre. Dans le cas particulier, où il s'agit d'une seule et même égalité

$$b = c \qquad (\text{gr. } \Omega)$$

répétée m fois, on aura

$$b^m = c^m \qquad (\text{gr. } \Omega),$$

c. à d. que l'on peut élever les deux membres d'une égalité à une puissance entière quelconque.

4.

Passons maintenant à la résolution du problème suivant. Etant donnés deux éléments quelconques a et b du groupe Ω, on demande de trouver un élément x tel qu'on ait

$$ax = b \qquad (\text{gr. } \Omega).$$

Nous allons montrer qu'il existe toujours un élément x satisfaisant à cette égalité et qu'il n'en existe qu'un seul. En effet, en composant l'élément a avec tous les éléments du groupe Ω on obtient m éléments différents, c. à d., abstraction faite de l'ordre, tous les éléments du groupe Ω. En particulier, la composition de a avec un certain élément u du groupe Ω donnera l'élément b. On voit qu'un tel élément u est unique. On aura donc

$$x = u \qquad (\text{gr. } \Omega).$$

Une égalité telle que

$$ax = b \qquad (\text{gr. } \Omega),$$

où l'inconnue x entre au premier degré porte le nom d'égalité de premier degré et u est la *solution* de cette égalité. Trouver la solution d'une égalité de premier degré se dit *résoudre* une telle égalité. On voit qu'une égalité de premier degré admet toujours une seule et unique solution. En particulier, l'égalité

$$ax = a \qquad (\text{gr. } \Omega)$$

admet une seule et unique solution

$$x = u \qquad (\text{gr. } \Omega).$$

Soit de même $x = v \qquad (\text{gr. } \Omega)$

la solution de l'égalité $bx = b \qquad (\text{gr. } \Omega)$

et $x = w \qquad (\text{gr. } \Omega),$

la solution de l'égalité $ax = b \qquad (\text{gr. } \Omega)$

on aura $bu = awu = aw = b \qquad (\text{gr. } \Omega),$

d'où $u = v \qquad (\text{gr. } \Omega).$

Le groupe Ω renferme donc un seul et unique élément u tel qu'en le composant avec tout élément du groupe Ω, on obtient ci dernier élément lui-même. L'élément u portera le nom d'*élément-unité* ou tout bonnement d'*unité* et sera désigné par 1.

5.

Soit a un élément quelconque du groupe Ω, en le composant avec lui-même on obtient a^2; de même en composant a^2 avec a on obtient a^3, et ainsi de suite. Or comme la suite

$$a, a^2, a^3, \ldots .$$

peut-être prolongée indéfiniment et que tous ses termes font partie du groupe Ω, elle doit renfermer des termes égaux; on aura, par exemple,

$$a^u = a^v \qquad (\text{gr. } \Omega),$$

où l'on peut supposer $u > v$. Il en résulte

$$a^{u-v} a^v = a^v \qquad (\text{gr. } \Omega),$$

d'où $a^{u-v} = 1 \qquad (\text{gr. } \Omega).$

Une certaine puissance de l'élément a est donc égale à l'unité. Soit t l'exposant le plus petit pour lequel on a

$$a^t = 1 \qquad (\text{gr. } \Omega),$$

il est clair d'abord que, quel que soit le nombre entier positif k, on aura

$$a^{kt} = 1 \qquad (\text{gr. } \Omega).$$

Je dis qu'inversement, si l'on a

$$a^u = 1 \qquad (\text{gr. } \Omega)$$

le nombre u est divisible par t. En effet, divisons u par t, si la division se fait sans reste, la proposition est démontrée; si non soit k le quotient et l le reste qu'on peut toujours supposer positif et inférieur à t. On aura

$$a^u = a^{kt+l} = 1 \qquad (\text{gr. } \Omega)$$

d'où

$$a^l = 1 \qquad (\text{gr. } \Omega),$$

contrairement à notre supposition que t est le plus petit exposant pour lequel on a

$$a^t = 1 \qquad (\text{gr. } \Omega).$$

Le nombre u est donc divisible par t. En général, si l'on a

$$a^u = a^v \qquad (\text{gr. } \Omega)$$

on en tirera

$$u \equiv v \qquad (\text{mod. } t)$$

et inversement.

Cela étant ainsi, on peut donner un sens aux puissances de a ayant pour exposant zéro ou un nombre entier négatif. En effet, on peut faire

$$a^0 = a^t = 1 \qquad (\text{gr. } \Omega).$$

De même si $-u$ est un nombre négatif divisible par t, on fera

$$a^{-u} = a^0 = 1 \qquad (\text{gr. } \Omega).$$

Si $-u$ n'est pas divisible par t, soit k le quotient et l le reste qu'on peut toujours supposer positif et inférieur à t; on fera

$$a^{-u} = a^{kt+l} = a^l \qquad (\text{gr. } \Omega).$$

Si l'on a

$$a^t = 1 \qquad (\text{gr. } \Omega)$$

et que t est le plus petit exposant positif pour lequel une telle égalité a lieu, on dit que, dans le groupe Ω, l'élément a appartient à l'exposant t. Tout élément du groupe Ω appartient donc à un certain exposant.

Appliquons les considérations précédentes à la résolution de l'égalité

$$ax = b \qquad (\text{gr. } \Omega).$$

Si a appartient à l'exposant t, la valeur

$$x = ba^{t-1} \qquad (\text{gr. } \Omega)$$

satisfait évidemment à l'égalité précédente. C'est donc la solution cherchée qui est unique comme nous le savons. Si l'on connaît l'exposant t[*] auquel appartient un élément a, il est facile de déterminer l'exposant auquel appartient une puissance de a telle que a^k, par exemple. En effet, soit d le plus grand commun diviseur de t et de k, on aura évidemment

$$a^{k\frac{t}{d}} = a^{t\frac{k}{d}} = 1 \qquad (\text{gr. } \Omega).$$

Si a^k appartenait à un exposant τ inférieur à $\frac{t}{d}$, on aurait

$$a^{k\tau} = 1 \qquad (\text{gr. } \Omega)$$

d'où

$$k\tau \equiv 0 \qquad (\text{mod. } t).$$

Le nombre $k\tau$ serait donc un multiple tant de k que de t, ce qui est impossible car on a

$$k\tau < k\frac{t}{d}$$

et $k\frac{t}{d}$ est le plus petit commun multiple de k et de t.

Si a appartient à l'exposant t et b à l'exposant u et t et u sont premiers entre eux, l'élément ab appartiendra à l'exposant tu. En effet on aura évidemment

$$(ab)^{tu} = a^{tu}b^{tu} = 1 \qquad (\text{gr. } \Omega).$$

Si ab appartenait à un exposant g inférieur à tu, on aurait

$$tu \equiv 0 \qquad (\text{mod. } g)$$

et le quotient $\frac{tu}{g}$ serait divisible par un nombre premier tel que p. D'où

$$(ab)^{\frac{tu}{p}} = 1 \qquad (\text{gr. } \Omega).$$

---

[*] Le cas particulier de cette proposition où t est une puissance de nombre premier nous suffirait.

Or un des nombres t et u et un seul seulement sera divisible par p. Que ce soit le nombre t; on aura

$$(ab)^{\frac{tu}{p}} = a^{\frac{t}{p}u} b^{\frac{t}{p}u} = a^{\frac{t}{p}u} = 1 \qquad (\text{gr. } \Omega).$$

D'où, en posant

$$uv \equiv 1 \qquad (\text{mod. } t),$$

$$a^{\frac{t}{p}uv} = a^{\frac{t}{p}} = 1 \qquad (\text{gr. } \Omega)$$

contrairement à la supposition que a appartient à l'exposant t.

6.

Soit Ω un groupe eulérien quelconque d'ordre n, si une partie des éléments du groupe Ω forment à eux-seuls un groupe* A, le groupe A est dit *sous-groupe* du groupe Ω. Il est clair que le groupe A sera aussi un groupe eulérien. Par extension nous considérons le groupe Ω comme un sous-groupe de lui-même. Soient A_1 et A_2 deux sous-groupes du groupe Ω d'ordres u_1 et u_2 respectivement et considérons l'expression

$$a_1 a_2 \qquad (\text{gr. } \Omega)$$

où a_1 désigne un élément du groupe A_1 et a_2 un élément du groupe A_2. L'ensemble de toutes les valeurs différentes que cette expression est susceptible de prendre formeront évidemment un groupe† B, car si $a_1' a_2'$ et $a_1'' a_2''$ sont deux valeurs de cette expression

$$a_1' a_2' a_1'' a_2'' = (a_1' a_1'')(a_2' a_2'') \qquad (\text{gr. } \Omega)$$

en sera une aussi. Dans le cas particulier où l'expression

$$a_1 a_2 \qquad (\text{gr. } \Omega)$$

est susceptible de prendre $u_1 u_2 = v$ valeurs différentes, c. à d. dans le cas où l'égalité

$$a_1' a_2' = a_1'' a_2'' \qquad (\text{gr. } \Omega)$$

n'a jamais lieu à moins qu'on n'ait en même temps

$$a_1' = a_1'' \qquad (\text{gr. } \Omega)$$

et

$$a_2' = a_2'' \qquad (\text{gr. } \Omega)$$

---

* On compose les éléments du groupe A comme s'ils faisaient partie du groupe Ω.

† Tous les groupes que nous considérons désormais seront des sous-groupes du groupe Ω.

nous dirons qu'on peut combiner les groupes A_1 et A_2 et nous écrirons

$$A_1A_2 = B \qquad (\text{gr. } \Omega).$$

Comme on a toujours

$$a_1a_2 = a_2a_1 \qquad (\text{gr. } \Omega)$$

l'expression A_2A_1 aura aussi un sens et nous aurons

$$A_2A_1 = A_1A_2 \qquad (\text{gr. } \Omega).$$

L'ordre de combinaison de deux groupes est donc indifférent. Si les groupes A_1 et A_2 ont en commun un élément a différent de l'élément-unité, il n'est pas possible de les combiner. En effet, on aura alors

$$a_1'a_2' = a_1''a_2'' \qquad (\text{gr. } \Omega),$$

en posant soit

$$a_1' = a \quad a_2' = 1 \qquad (\text{gr. } \Omega),$$

soit

$$a_1'' = 1 \quad a_2'' = a \qquad (\text{gr. } \Omega).$$

Je dis que si les groupes A_1 et A_2 n'ont de commun que l'élément-unité, il est toujours possible de les combiner. En effet, si l'on avait

$$a_1'a_2' = a_1''a_2'' \qquad (\text{gr. } \Omega)$$

sans avoir en même temps

$$a_1' = a_1'' \qquad (\text{gr. } \Omega),$$
$$a_2' = a_2'' \qquad (\text{gr. } \Omega),$$

on poserait

$$a_1''a_1''' = a_1' \qquad (\text{gr. } \Omega),$$
$$a_2'a_2''' = a_2'' \qquad (\text{gr. } \Omega),$$

où a_1''' et a_2''' feraient partie des groupes A_1 et A_2 respectivement et ne seraient pas en même temps identiques à l'élément-unité. L'égalité

$$a_1'a_2' = a_1''a_2'' \qquad (\text{gr. } \Omega)$$

entraînerait alors la suivante:

$$a_1''a_1'''a_2' = a_1''a_2'a_2''' \qquad (\text{gr. } \Omega)$$

d'où

$$a_1''' = a_2''' \qquad (\text{gr. } \Omega),$$

contrairement à la supposition que les groupes A_1 et A_2 n'ont de commun que l'élément-unité.

Soient maintenant $A_1, A_2, \dots . A_m$

m groupes d'ordres $u_1, u_2, \dots . u_m$

respectivement, s'il est possible de combiner les groupes A_1 et A_2, puis les groupes A_1A_2 et A_3 et ainsi de suite jusqu'au dernier groupe A_m, nous dirons qu'il est possible de combiner les groupes

$$A_1,\ A_2,\ \ldots\ldots A_m$$

dans l'ordre indiqué et si le résultat qu'on obtient est un groupe B, nous écrirons

$$A_1A_2 \ldots\ldots A_m = B \qquad (\text{gr. } \Omega).$$

Il est clair que le groupe B sera d'ordre

$$v = u_1u_2 \ldots\ldots u_m.$$

Soit d'une manière générale a_h un élément du groupe A_h, l'expression

$$a_1a_2 \ldots\ldots a_m \qquad (\text{gr. } \Omega)$$

est alors susceptible deprendre v valeurs différentes et l'on n'aura jamais

$$a_1'a_2' \ldots\ldots a_m' = a_1''a_2'' \ldots\ldots a_m'' \qquad (\text{gr. } \Omega)$$

à moins d'avoir en même temps

$$\begin{array}{ll} a_1' = a_1'' & (\text{gr. } \Omega), \\ a_2' = a_2'' & (\text{gr. } \Omega), \\ \ldots\ldots\ldots & \ldots\ldots \\ a_m' = a_m'' & (\text{gr. } \Omega). \end{array}$$

Cela est clair d'abord si l'on fait prendre à a_1a_2 toutes ses valeurs, puis à $(a_1a_2)\,a_3$ toutes ses valeurs, et ainsi de suite jusqu'au dernier a_m. Or nous avons vu qu'il y a plusieurs manières de réunir les éléments

$$a_1,\ a_2 \ldots\ldots a_m$$

en un seul et que le résultat sera le même de quelque manière que l'on procède. C'est ainsi que de l'expression

$$a_1a_2 \ldots\ldots a_m \qquad (\text{gr. } \Omega)$$

on peut déduire une autre en réunissant les éléments a_h et a_k en un seul élément a_ha_k. L'expression ainsi déduite ne sera qu'un produit de $m-1$ éléments. En faisant prendre à a_ha_k toutes ses valeurs qui seront au nombre de u_hu_k, on obtient tous les éléments du groupe $A_h A_k$. En opérant de la même manière sur la suite de $m - 1$ éléments à laquelle on parvient en remplaçant dans la suite

$$a_1, a_2, \ldots . a_m$$

les éléments a_h et a_k par leur produit $a_h a_k$, on obtient une nouvelle suite de $m - 2$ éléments et ainsi de suite jusqu'à ce que tous les éléments

$$a_1, a_2, \ldots . a_m$$

se trouvent réunis en un seul. A chacune de ces manières de réunir les éléments

$$a_1, a_2, \ldots . a_m$$

en un seul, correspond une manière de réunir les groupes

$$A_1, A_2, \ldots . A_m$$

en un seul; les opérations seront toujours exécutables et l'on parviendra au même résultat. Si donc il est possible de combiner les groupes

$$A_1, A_2, \ldots . A_m$$

d'une certaine manière, il est aussi possible de les combiner de toute autre manière et le résultat sera toujours le même. Nous avons donc deux critères pour juger de la possibilité de réunir les groupes

$$A_1, A_2, \ldots . A_m$$

en un seul; le premier est de voir s'il est possible de les réunir en un seul en procédant d'une certaine manière, le second c'est de s'assurer si l'expression

$$a_1 a_2 \ldots . a_m \qquad (\text{gr. } \Omega)$$

admet

$$v = u_1 u_2 \ldots . u_m$$

valeurs différentes.

Un sous-groupe A qu'il est possible d'obtenir par la combinaison de deux ou plusieurs sous-groupes est dit *groupe combiné.* Au contraire tout sous-groupe qu'il est impossible d'obtenir par la combinaison de deux ou plusieurs sous-groupes est dit *groupe simple.* Soit A un sous-groupe quelconque d'ordre m et a un élément de ce groupe. Si a appartient à un exposant t, les puissances de a donneront t éléments différents

$$1, a, a^2, a^3, \ldots . a^{t-1}$$

et pas plus. Tous ces éléments font partie du groupe A, donc $t \leqq m$. Dans le cas particulier où $t = m$, tous les éléments du groupe A sont des puissances de a qui porte alors le nom de *base*. Tout élément du groupe A qui peut servir de base est dit *racine primitive*. Tout groupe qui admet des racines primitives est dit *monobase*, tandis qu'un groupe pour lequel il n'existe pas de racines primitives est dit *polybase*. Le sens étymologique de ce mot sera justifié plus tard; pour le moment nous désignerons sous le nom de *polybase* tout groupe qui n'est pas monobase.

Soit a un élément du groupe Ω appartenant à un exposant t égal à une puissance de nombre premier s^u, les éléments

$$1, a, a^2, \ldots. a^{s^u-1}$$

formeront un groupe monobase A d'ordre s^u. Je dis que le groupe A est un groupe simple. En effet, si A était décomposable en un produit* de plusieurs

groupes $$A_1, A_2, \ldots. A_h,$$

on aurait $$A = A_1 A_2 \ldots. A_h \qquad (\text{gr. } \Omega).$$

Tout groupe du second membre doit renfermer l'élément-unité et par suite tous les éléments d'un groupe tel que A_k font nécessairement partie du groupe A. Si le groupe A_k renfermait un élément a^ι du groupe A appartenant à l'exposant s^u, il renfermerait aussi les éléments

$$1, a^\iota, a^{2\iota}, \ldots. a^{\iota(s^u-1)}$$

qui sont tous différents entre eux. Le groupe A_k serait alors d'ordre s^u et les autres facteurs seraient identiques au *groupe-unité*, c. à d. à l'élément-unité considéré comme un groupe. Le groupe A_k ne pouvant renfermer aucun élément appartenant à l'exposant s^u, tous ses éléments appartiendront à un diviseur de s^{u-1}. Il en sera évidemment de même de tout élément du produit

$$A_1 A_2 \ldots. A_h.$$

Comme le groupe A, renferme des éléments appartenant à l'exposant s^u, il ne peut être identique au produit

$$A_1 A_2 \ldots. A_h.$$

Tout groupe monobase dont l'ordre est une puissance de nombre premier est donc un groupe simple.

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* Nous empruntons encore la terminologie de la multiplication.

7.

Passons maintenant à la résolution de l'égalité

$$x^{s^u} = 1 \qquad (\text{gr. } \Omega)$$

où s est un nombre premier et u un nombre entier positif. L'ensemble de toutes les solutions de cette égalité formeront un groupe que nous désignerons par Ξ_u. En effet, si l'on a

$$a^{s^u} = 1 \qquad (\text{gr. } \Omega)$$

et

$$b^{s^u} = 1 \qquad (\text{gr. } \Omega),$$

on aura aussi

$$(ab)^{s^u} = 1 \qquad (\text{gr. } \Omega).$$

Considérons d'abord l'égalité

$$x^s = 1 \qquad (\text{gr. } \Omega).$$

Les solutions de cette égalité se composent de l'élément-unité et des éléments appartenant à l'exposant s. Si le groupe Ξ_1 ne renferme aucun élément appartenant à l'exposant s, on aura évidemment

$$\Xi_1 = U \qquad (\text{gr. } \Omega),$$

où U désigne le groupe-unité. Si, au contraire, le groupe Ξ_1 renferme un élément a_1 appartenant à l'exposant s, les puissances

$$1, a_1, a_1^2, \ldots . a_1^{s-1},$$

formeront un groupe monobase A_1^s d'ordre s qui sera évidemment un sous-groupe du groupe Ξ_1. Si le groupe Ξ_1 ne renferme aucun élément en dehors de ceux du groupe A_1^s, on aura

$$\Xi_1 = A_1^s.$$

Si non, soit a_2 un élément du groupe Ξ_1 différent de ceux du groupe A_1^s, les puissances

$$1, a_2, a_2^2, \ldots . a_2^{s-1}$$

formeront un groupe monobase d'ordre s qui n'aura de commun avec le groupe A_1^s que l'élément-unité. En effet, si un élément du groupe A_2^s autre que l'élément-unité faisait partie du groupe A_1^s, tout le groupe A_2^s ferait partie du groupe A_1^s, contrairement à la supposition que l'élément a_2 ne fait pas partie du groupe A_1^s. De même, si le groupe Ξ_1 renferme un élément a_3 ne figurant ni parmi ceux du groupe A_1^s ni parmi ceux du groupe A_2^s, les puissances

$$1, a_3, a_3^2, \ldots . a_3^{s-1}$$

formeront un groupe monobase A_3^s qui n'aura de commun avec les groupes A_1^s et A_2^s que l'élément-unité. En continuant de la même manière, on finira par former un certain nombre h de groupes monobases d'ordre s tels que deux quelconques de ces groupes n'auront de commun que l'élément-unité et que tous les h groupes pris ensemble renfermeront tous les éléments du groupe Ξ_1 et même chaque élément une seule fois, si l'on convient de ne compter qu'une seule fois l'élément-unité qui fait partie nécessairement de tout sous-groupe de Ω. Soit ξ_1 l'ordre du groupe Ξ_1, nous aurons

$$\xi_1 = h(s-1) + 1.$$

Soit maintenant a_1 un élément du groupe Ξ_1 différent de l'élément-unité, les éléments

$$1,\ a_1,\ a_1^2,\ \ldots .\ a_1^{s-1}$$

formeront un groupe monobase A_1^s d'ordre s. Si a_2 est un élément du groupe Ξ_1 ne faisant pas partie du groupe A_1^s, le groupe monobase A_2^s formé par les éléments

$$1,\ a_2,\ a_2^2,\ \ldots .\ a_2^{s-1}$$

n'aura de commun avec le groupe A_1^s que l'élément-unité. Posons donc

$$A_{1,2}^{s^2} = A_1^s A_2^s \qquad (\text{gr. } \Omega),$$

le groupe $A_{1,2}^{s^2}$ d'ordre s^2 sera évidemment un sous-groupe du groupe Ξ_1. Si le groupe Ξ_1 ne renferme pas d'autres éléments que ceux du groupe $A_{1,2}^{s^2}$, on aura

$$\Xi_1 = A_1^s A_2^s \qquad (\text{gr. } \Omega).$$

Dans le cas contraire, soit a_3 un élément du groupe Ξ_1 ne faisant pas partie du groupe $A_{1,2}^{s^2}$, le groupe monobase A_3^s formé par les éléments

$$1,\ a_3,\ a_3^2,\ \ldots .\ a_3^{s-1}$$

n'aura de commun avec le groupe $A_{1,2}^{s^2}$ que l'élément-unité. Autrement le groupe A_3^s serait un sous-groupe du groupe $A_{1,2}^{s^2}$ et l'élément a_3 ferait partie du groupe $A_{1,2}^{s^2}$ contrairement à notre supposition. Posons

$$A_{1,2,3}^{s^3} = A_1^s A_2^s A_3^s \qquad (\text{gr. } \Omega),$$

$A_{1,2,3}^{s^3}$ sera un sous-groupe de Ξ_1. Si l'on n'a pas

$$\Xi_1 = A_{1,2,3}^{s^3} \qquad (\text{gr. } \Omega),$$

soit a_4 un élément du groupe Ξ_1 ne faisant pas partie du groupe $A_{1,2,3}^{s^3}$, etc. En continuant de la même manière on finira par trouver une décomposition telle que

$$\Xi_1 = A_1^s A_2^s \ldots . A_{m_1}^s \qquad (\text{gr. } \Omega),$$

d'où

$$\xi_1 = s^{m_1}$$

et par suite

$$h = \frac{s^{m_1} - 1}{s - 1}.$$

Comme tous les éléments du groupe Ξ_1 satisfont à l'égalité

$$x^s = 1 \qquad (\text{gr. } \Omega),$$

il est clair que tout sous-groupe monobase du groupe Ξ_1 sera d'ordre s, saut le groupe-unité. Toute décomposition de Ξ_1 en un produit de groupes monobases contiendra donc m_1 groupes monobases d'ordre s. Nous exprimerons ce fait, en disant que la décomposition de Ξ_1 en groupes monobases est univoque quant au nombre de ces groupes monobases composants d'un ordre déterminé. Tout sous-groupe monobase de Ξ_1 est simple. En effet, un tel groupe—abstraction faite du groupe-unité—est d'ordre s et par suite il est simple comme tout groupe monobase dont l'ordre est une puissance de nombre premier. Tout sous-groupe simple χ du groupe Ξ_1 est monobase. En effet, un tel groupe χ est identique à l'ensemble des solutions de l'égalité

$$x^s = 1 \qquad (\text{gr. } \chi)$$

et sera par conséquent égal à un groupe monobase.

Il est facile de déterminer le nombre des décompositions de Ξ_1 en groupes monobases. En effet pour A_1 il y a choix entre $\frac{s^{m_1} - 1}{s - 1}$ groupes. Pour A_2 il y aura choix entre $\frac{s^{m_1} - 1}{s - 1} - 1$ groupes. Le groupe $A_{1,2}$ étant identique à l'ensemble des solution de l'égalité

$$x^s = 1 \qquad (\text{gr. } A_{1,2}),$$

renfermera $\frac{s^2 - 1}{s - 1}$ groupes monobases d'ordre s. Pour A_3 il y aura donc choix entre $\frac{s^{m_1} - s^2}{s - 1}$ groupes. De même, pour A_4 il y aura choix entre $\frac{s^{m_1} - s^3}{s - 1}$ groupes et ainsi de suite jusqu'au groupe A_{m_1} pour lequel il y aura choix entre $\frac{s^{m_1} - s^{m_1 - 1}}{s - 1}$

groupes. Le nombre des décompositions de Ξ_1 en groupes monobases est donc égal à

$$\frac{s^{m_1}-1}{s-1}\cdot\frac{s^{m_1}-s}{s-1}\cdot\frac{s^{m_1}-s^2}{s-1}\cdots\cdot\frac{s^{m_1}-s^{m_1-1}}{s-1}:m_1!$$

puisque nous ne considérerons comme distinctes les décompositions qui ne diffèrent que par l'ordre de leurs facteurs.

Il est clair que tout élément du groupe Ξ_1 n'est représentable que d'une seule manière comme un produit d'éléments appartenant respectivement aux groupes

$$A_1,\ A_2,\ \ldots.\ A_{m_1}$$

si l'on fixe ces derniers groupes. En particulier, si l'on veut représenter l'élément-unité comme un produit d'éléments appartenant respectivement aux groupes

$$A_1,\ A_2,\ \ldots.\ A_{m_1},$$

il faut prendre l'élément-unité dans chacun de ces groupes.

Nous pouvons rechercher maintenant le nombre des solutions d'un égalité telle que

$$x^s = a \qquad (\text{gr. } \Omega).$$

En effet, soient

$$1,\ c_1,\ c_2,\ \ldots.\ c_{s^{m_1}-1}$$

les s^{m_1} solutions de l'égalité*

$$x^s = 1 \qquad (\text{gr. } \Omega),$$

si l'égalité

$$x^s = a \qquad (\text{gr. } \Omega)$$

admet une solution b_0, les éléments

$$b_0,\ b_0c_1,\ b_0c_2,\ \ldots.\ b_0c_{s^{m_1}-1}$$

seront encore des solutions de la même égalité différentes entre elles. Inversement, si b_h est une solution de l'égalité

$$x^s = a \qquad (\text{gr. } \Omega),$$

l'élément $b_0^{-1}b_h$ satisfera à l'égalité

$$x^s = 1 \qquad (\text{gr. } \Omega)$$

et sera par conséquent identique à une solution c_h de cette égalité, d'où

$$b_h = b_0ch \qquad (\text{gr. } \Omega).$$

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* Nous supposons que le groupe Ω contient des éléments appartenant à l'exposant s.

On peut donc faire correspondre à chaque solution de l'égalité

$$x^{s} = a \qquad (\text{gr. } \Omega)$$

une solution bien déterminé de l'égalité

$$x^{s} = 1 \qquad (\text{gr. } \Omega),$$

de manière qu'à deux solutions différentes de l'égalité

$$x^{s} = a \qquad (\text{gr. } \Omega)$$

correspondent deux solutions différentes de l'égalité

$$x^{s} = 1 \qquad (\text{gr. } \Omega)$$

et inversement. Le nombre des solutions de l'égalité

$$x^{s} = a \qquad (\text{gr. } \Omega)$$

est donc égal à s^{m_1} dans le cas où elle en a aumoins une. Il est facile de prouver par des exemples que le cas où l'égalité

$$x^{s} = a \qquad (\text{gr. } \Omega)$$

n'admet aucune solution se présente aussi.

8.

Nous pouvons maintenant aborder l'étude du groupe Ξ_u, c. à d. de l'ensemble des solutions de l'égalité

$$x^{s^u} = 1 \qquad (\text{gr. } \Omega)$$

où s est un nombre premier et u un nombre entier quelconque. Si s^θ est la puissance la plus élevée de s à laquelle comme exposant peut appartenir un élément du groupe Ω, nous dirons que le groupe Ω est de *rang* θ par rapport au nombre premier s. Dans le cas où le groupe Ω ne renferme aucun élément appartenant à une puissance de s, nous dirons qu'il est de rang θ par rapport audit nombre premier s. Le groupe Ξ_θ se réduit alors à un élément unique, à l'élément-unité. Laissons donc de côté ce cas, c. à d. supposons $\theta > 0$, et proposons nous d'étudier le groupe Ξ_θ. Il est clair d'ailleurs que pour toute valeur de u égale ou supérieure à θ on aura

$$\Xi_u = \Xi_\theta .$$

Nous pouvons donc nous borner à la considération des groupes Ξ_u où $u \leqq \theta$. Soit Ξ_u un tel groupe, élevons tous ses éléments à la puissance s; les éléments différents qu'on obtiendra de cette manière formeront à eux-seuls un groupe eulérien que nous désignerons par Ξ_u et qui sera évidemment un sous-groupe de Ξ_u. De même, nous désignerons par Ξ_u'' le groupe formé par l'ensemble des éléments différents qu'on obtient par l'élevation de tous les éléments du groupe Ξ_u à la puissance s^2, et ainsi de suite jusqu'au groupe $\Xi_u^{(u)}$ qui se réduira évidemment au groupe-unité. Cela étant ainsi, je dis que le groupe H_k^u formé par l'ensemble des solutions de l'égalité

$$x^s = 1 \qquad (\text{gr. } \Xi_u^{(k-1)})$$

est indépendant de u pourvu qu'on ait

$$u \geqq k.$$

En effet, un groupe tel que Ξ_{u+1} où $u+1 \leqq \theta$, ne diffère du groupe Ξ_u que par les éléments appartenant à l'exposant s^{u+1} que le premier a en plus sur le dernier. Le groupe $\Xi_{u+1}^{(k-1)}$ ne différera du groupe $\Xi_u^{(k-1)}$ que par les éléments appartenant à l'exposant s^{u-k+2} que le premier aura en plus sur le second. Or comme on a

$$u - k + 2 > 1,$$

ces éléments n'entrerons pas dans le groupe H_k^{u+1} et par suite on aura

$$H_k^{u+1} = H_k^u.$$

Nous pouvons donc écrire désormais H_k au lieu de H_k^u. Ce point admis, je désigne respectivement par

$$s^{m_1},\ s^{m_2},\ \ldots.\ s^{m_\theta}$$

les ordres des groupes $\qquad H_1,\ H_2,\ \ldots.\ H_\theta$

et je dis qu'on aura

$$m_1 \geqq m_2 \geqq m_3 \geqq \ldots. \geqq m_{\theta-1} \geqq m_\theta > 0.$$

En effet, Ξ_k' est un sous-groupe de Ξ_k, et comme Ξ_k' ne renferme pas d'éléments appartenant à l'exposant s^k, Ξ_k' est aussi un sous-groupe de Ξ_{k-1}, et par suite $\Xi_k^{(k-1)}$ est un sous-groupe de $\Xi_{k-1}^{(k-2)}$ et H_k est un sous-groupe de H_{k-1}; donc $m_{k-1} \geqq m_k$. L'inégalité

$$m_\theta > 0$$

résulte de ce que le groupe Ξ_θ renferme, d'après la supposition, au moins un élément appartenant à l'exposant s^θ, et par suite le groupe $\Xi_\theta^{(\theta-1)}$ ou H_θ renfermera au moins un élément appartenant à l'exposant s; son ordre sera donc supérieur à l'unité.

Soit A^{s^u} un sous-groupe monobase Ξ_θ d'ordre s^u et a_u une racine primitive du groupe A^{s^u}. Le groupe A^{s^u} se composera des éléments

$$1,\ a_u,\ a_u^2,\ \ldots .\ a^{s^u-1}.$$

Soit maintenant s^k une puissance de s, non supérieure à s^u, l'ensemble des éléments du groupe A^{s^u} appartenant à des exposants non supérieurs à s^k sera donné par les puissances suivantes

$$1,\ a^{s^{u-k}},\ a^{2s^{u-k}},\ \ldots .\ a^{(s^k-1)s^{u-k}}.$$

On voit que ces éléments sont au nombre de s^k et qu'ils constituent un groupe monobase A^{s^k} à eux-seuls. A^{s^k} est d'ailleurs le seul sous-groupe de A^{s^u} d'ordre s^k, car tous les éléments du groupe A^{s^u} qui ne font pas partie du groupe A^{s^k}, appartiennent à des exposants supérieurs à s^k. Un tel groupe A^{s^k} portera le nom d'*émettant* d'ordre s^k du groupe A^{s^u}. On voit que le groupe A^{s^u} n'a qu'un seul émettant d'un ordre déterminé s^k non supérieur à s^u. Le groupe A^{s^u} porte le nom d'*émanant* d'ordre s^u du groupe A^{s^k}. Un sous-groupe monobase A^{s^k} du groupe Ξ_θ étant donné et une certaine puissance s^u non inférieure à s^k et non supérieure à s^θ étant fixée, trois cas pourront se présenter: 1) le groupe A^{s^k} n'aura aucun émanant d'ordre s^u; 2) il n'en aura qu'un seul; 3) il en aura plusieurs. Un sous-groupe monobase de Ξ_θ est dit de *portée* (range, Tragweite) s^u s'il admet au moins un émanant d'ordre s^u et n'en admet aucun d'ordre s^{u+1}. La portée d'un groupe dépend naturellement du groupe Ξ_θ, car si un groupe monobase A^{s^k} où $k < \theta$ est de portée s^θ dans le groupe Ξ_θ, il ne peut être que de portée $s^{\theta-1}$ dans le groupe $\Xi_{\theta-1}$. Tous nos groupes seront d'ailleurs des sous-groupes de Ξ_θ à moins que nous n'avertissions du contraire. Cela étant ainsi, voici le problème qu'il s'agit de résoudre: Soit A^{s^k} un groupe monobase d'ordre s^k inférieur à s^θ et de portée s^u supérieure à s^k, on demande le nombre de ses émanants d'ordre s^{k+1} et de portées s^u, s^{u-1}, $\ldots .\ s^{k+1}$ respectivement. Il est clair que le groupe A^{s^k} ne pourra avoir d'autres émanants d'ordre s^{k+1} que ceux que nous venons de mentioner.

Soit donc A^{s^k} un groupe monobase d'ordre s^k et de portée s^u et a_k une racine primitive de ce groupe. D'après la supposition, il existe un groupe monobase A^{s^u} d'ordre s^u ayant A^{s^k} pour émettant. Soit a_u une racine primitive de A^{s^u}, a_k sera égal à une puissance telle que $a_u^{gs^{u-k}}$ où g n'est pas divisible par s. L'élément a_k est donc égal à une puissance s^{u-k} d'un élément appartenant à l'exposant s_u et fait par conséquent partie du groupe $\Xi_u^{(u-k)}$. Soit maintenant $A^{s^{k+1}}$ un émanant[*] de A^{s^k} de portée s^u et a_{k+1} une racine primitive de $A^{s^{k+1}}$, a_k sera évidemment égal à a_{k+1}^{hs} où h n'est pas divisible par s. Or a_{k+1}^h étant aussi une racine primitive de $A^{s^{k+1}}$, on peut remplacer a_{k+1}^h par a_{k+1} de sorte que a_k soit égal à a_{k+1}^s. Comme le groupe $A^{s^{k+1}}$ est de portée s^u, sa racine primitive a_{k+1} fera partie du groupe $\Xi_u^{(u-k-1)}$. D'ailleurs a_k fait aussi partie du groupe $\Xi_u^{(u-k-1)}$, à cause de ce que $\Xi_u^{(u-k)}$ est un sous-groupe de $\Xi_u^{(u-k-1)}$. Cela étant ainsi, tout émanant $A^{s^{k+1}}$ (du groupe A^{s^k}) d'ordre s^{k+1} et de portée s^u peut être considéré comme engendré par une base a_{k+1} satisfaisant à l'égalité

$$x^s = a_k \qquad (\text{gr. } \Xi_u^{(u-k-1)}).$$

Inversement, si b_{k+1} est une solution de cette égalité, b_{k+1} appartiendra à l'exposant s^{k+1} et les éléments

$$1,\ b_{k+1},\ b_{k+1}^2,\ \ldots.\ b_{k+1}^{s^{k+1}-1}$$

formeront un groupe $B^{s^{k+1}}$ d'ordre s^{k+1}. Le groupe $B^{s^{k+1}}$ aura évidemment pour sous-groupe le groupe A^{s^k} car ce dernier se compose des éléments

$$1,\ b_{k+1}^s,\ b_{k+1}^{2s},\ \ldots.\ b_{k+1}^{(s^k-1)s}.$$

D'autre part le groupe $B^{s^{k+1}}$ est de portée s^u tout au moins en tant que b_{k+1} fait partie de $\Xi_u^{(u-k-1)}$ et par suite est égal à une puissance s^{u-k-1} d'un élément appartenant à l'exposant s^u. Le groupe $B^{s^{k+1}}$ ne peut être d'une portée supérieure à s^u; autrement le groupe A^{s^k} le serait aussi; donc $B^{s^{k+1}}$ est un émanant (du groupe A^{s^k}) d'ordre s^{k+1} et de portée s^u.

L'égalité $$x^s = a_k \qquad (\text{gr. } \Xi_u^{(u-k-1)})$$

admet nécessairement la solution

$$x = (a_u^{gs^{u-k-1}})\dagger \qquad (\text{gr. } \Xi_u^{(u-k-1)}).$$

Les solutions de cette égalité seront donc au nombre de $s^{m_{u-k}}$; nous les désignerons par

$$c_1,\ c_2,\ \ldots.\ c_{s^{m_{u-k}}}.$$

Or si la base c_1 engendre un émanant $A^{s^{k+1}}$ (du groupe A^{s^k}) d'ordre s^{k+1} et de portée s^u, les bases

$$c_1^{s^k+1},\ c_1^{2s^k+1},\ \ldots.\ c_1^{(s-1)s^k+1}$$

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[*] L'émettant d'ordre s^k+1 du groupe A^{s^u} est un tel émanant du groupe A^{s^k}.

[†] Les parenthèses veulent dire que l'élément $a_u^{gs^{u-k-1}}$ obtenu explicitement fait partie du groupe $\Xi_u^{(u-k-1)}$.

engendreront le même groupe $A^{s^{k+1}}$ et satisferont à l'égalité

$$x^s = a_k \qquad (\text{gr. } \Xi_u^{(u-k-1)})$$

et en dehors de la base c_1, ce seront les seules bases de $A^{s^{k+1}}$ qui jouiront de cette propriété. Il y aura donc s solutions de l'égalité précédente pour chaque émanant tel que $A^{s^{k+1}}$. Le nombre des émanants (du groupe A^{s^k}) d'ordre s^{k+1} et de portée s^u est donc égal à $s^{m_{u-k}-1}$. On prouvera de la même manière que le nombre des émanants (du groupe A^{s^k}) d'ordre s^{k+1} et de portée s^u *tout au moins* est égal à $s^{m_{u-k-1}-1}$. Le nombre des émanants (du groupe A^{s^k}) d'ordre s^{k+1} et de portée s^{u-1} est donc égal à

$$s^{m_{u-k-1}-1} - s^{m_{u-k}-1}$$

nombre qui peut se réduire quelquefois à zéro. D'une manière générale le nombre des émanants (du groupe A^{s^k}) d'ordre s^{k+1} et de portée s^{u-i} où

$$u > u - i \geqq k + 1$$

est égal à

$$s^{m_{u-k-i}-1} - s^{m_{u-k-i+1}-1}$$

nombre qui peut se réduire quelquefois à zéro.

Cherchons maintenant le nombre des groupes monobases d'ordre s et de portées s^θ, $s^{\theta-1}$, $s^{\theta-2}$, ... s^2, s respectivement car il ne peut y en avoir d'autres. Soit A^s un groupe monobase d'ordre s et de portée s^θ et a_1 une racine primitive de ce groupe. Soit A^{s^θ} un émanant (du groupe A^s) d'ordre s^θ et a_θ une racine primitive du groupe A^{s^θ}. L'élément a_1 sera égal à une puissance de a_θ telle que $a_\theta^{g s^{\theta-1}}$ où g n'est pas divisible par s. L'élément a_1 est donc une puissance $s^{\theta-1}$ d'un élément a_θ^g appartenant à l'exposant s^θ; il s'ensuit que l'élément a_1 fait partie du groupe $\Xi_\theta^{(\theta-1)}$ ou du groupe H_θ qui lui est identique. Le groupe A^s est donc un sous-groupe du groupe H_θ. Inversement tout sous-groupe monobase d'ordre s du groupe H_θ est de portée s^θ. En effet, soit B^s un tel sous-groupe, b_1 une de ses racines primitives et b_θ une solution de l'égalité

$$x^{s^{\theta-1}} = b_1 \qquad (\text{gr. } \Xi_\theta)$$

solution qui existe nécessairement, car b_1 fait partie du groupe $\Xi_\theta^{(\theta-1)}$. L'élément b_θ pourra servir de base à un groupe B^{s^θ} d'ordre s^θ qui sera un émanant du groupe B^s. Le groupe B^s est donc de portée s^θ au moins. Or comme aucun groupe ne peut avoir une portée supérieure à s^θ, il en résulte que B^s est de portée s^θ. Donc tout groupe monobase d'ordre s et de portée s^θ est un sous-groupe de H_θ et inversement tout sous-groupe monobase de H_θ d'ordre s est de portée s^θ. Le nombre de tels groupes est donc égal à

$$\frac{s^{m_\theta}-1}{s-1}.$$

De même, tout groupe monobase d'ordre s et de portée $s^{\theta-1}$ est un sous-groupe de $H_{\theta-1}$ et tout sous-groupe monobase de $H_{\theta-1}$ d'ordre s est de portée $s^{\theta-1}$ au moins. Le nombre des groupes monobases d'ordre s et de portée $s^{\theta-1}$ au moins est donc égal à

$$\frac{s^{m_{\theta-1}}-1}{s-1}.$$

Le nombre des groupes monobases d'ordre s et de portée $s^{\theta-1}$ est donc égal à

$$\frac{s^{m_{\theta-1}}-s^{m_\theta}}{s-1}$$

nombre qui se réduit quelquefois à zéro. D'une manière analogue, on trouvera que le nombre des groupes monobases d'ordre s et de portée s^u, où $u<\theta$, est égal à

$$\frac{s^{m_u}-s^{m_{u+1}}}{s-1}$$

nombre qui se réduit quelquefois à zéro.

Rangeons donc tous les groupes monobases d'ordre s en classes suivant qu'ils sont de portée s^θ, $s^{\theta-1}$, ... s^2 ou s. Faisons émaner de chaque groupe d'ordre s et de portée s^θ ses émanants d'ordre s^2 distribués en classes suivant qu'ils sont de portée s^θ, $s^{\theta-1}$, ... s^3 ou s^2. Ces émanants seront respectivement au nombre de

$$s^{m_{\theta-1}-1},\ s^{m_{\theta-2}-1}-s^{m_{\theta-1}-1},\ \ldots.\ s^{m_3-1}-s^{m_4-1},\ s^{m_2-1}-s^{m_3-1},\ s^{m_1-1}-s^{m_2-1}.$$

De même faisons émaner de chaque groupe monobase d'ordre s et de portée s^u où $u<\theta$, ses émanants d'ordre s^2 distribués en classes suivant qu'ils sont de portées s^u, s^{u-1}, ... s^3, s^2 respectivement. Ces émanants seront respectivement au nombre de

$$s^{m_{u-1}-1},\ s^{m_{u-2}-1}-s^{m_{u-1}-1},\ \ldots.\ s^{m_2-1}-s^{m_3-1},\ s^{m_1-1}-s^{m_2-1}.$$

Faisons de même émaner de tout groupe d'ordre s^2 ainsi obtenu et de portée s^u, où $3\leqq u\leqq\theta$, tous ses émanants d'ordre s^3 distribués en classes suivant qu'ils sont de portée s^u, s^{u-1}, ... ou s^3. Ces émanants seront respectivement au nombre de

$$s^{m_{u-2}-1},\ s^{m_{u-3}-1} - s^{m_{u-2}-1},\ \ldots\ s^{m_3-1} - s^{m_2-1},\ s^{m_1-1} - s^{m_2-1}.$$

Continuons la même opération jusqu'à ce qu'on soit parvenu aux groupes monobases d'ordre s^θ. Enfin, on peut faire émaner du groupe-unité, les groupes monobases d'ordre s rangés en classes suivant qu'ils sont de portée s^θ, $s^{\theta-1}$, ... ou s respectivement. Nous avons déjà déterminé le nombre de ces *émanants* du groupe-unité. Le groupe-unité pourra être considéré comme émettant d'ordre s^θ de tout groupe monobase d'ordre s^u. Tout groupe monobase d'ordre s^u trouvera nécessairement une place dans cette classification, car il a ces émettants bien déterminés d'ailleurs d'ordres s^{u-1}, s^{u-2}, ... s^2, s et s^0. Les éléments que deux groupes monobases A^{s^u} et A^{s^v} d'ordres s^u et s^v ont en commun, constituent un groupe monobase à eux-seuls. En effet, en laissant de côté le cas où les groupes A^{s^u} et A^{s^v} n'ont en commun que l'élément-unité qui peut-être considéré comme constituant un groupe monobase à lui-seul, soit a un élément faisant partie tant du groupe A^{s^u} que du groupe A^{s^v} et s^w l'exposant auquel a appartient. On peut supposer que les groupes A^{s^u} et A^{s^v} n'ont en commun aucun élément appartenant à un exposant supérieur à s^w. Cela étant ainsi, tous les éléments du groupe monobase A^{s^w} formé par les puissances

$$1,\ a,\ a^2 \ldots a^{s^w-1}$$

feront partie tant de A^{s^u} que de A^{s^v} et ces derniers groupes n'auront en commun aucun élément ne faisant pas partie du groupe A^{s^w}, car le groupe A^{s^w} renferme tous les éléments des groupes A^{s^u} et A^{s^v} qui appartiennent à un exposant non supérieur à s^w. Un tel tableau de classification de tous les groupes monobases de Ξ_θ est susceptible d'une représentation géométrique. En effet, menons une ligne d'une longueur égale à l'unité pour répresenter le groupe-unité et faisons émaner de cette ligne des faisceaux de lignes de longueur $s - 1$, une pour chaque groupe d'ordre s de manière qu'il y ait un faisceau séparé pour chaque portée. Une telle ligne représentera les éléments d'un groupe s qui appartiennent à l'exposant s. De chaque ligne de longueur $s - 1$ faisons émaner des faisceaux de lignes de longueur $s^2 - s$, une pour chaque groupe d'ordre s^2 et de manière qu'il y ait un faisceau séparé pour chaque portée. Une telle ligne de longueur $s^2 - s$ représentera les éléments d'un groupe d'ordre s^2 qui appartiennent à l'exposant s^2. De même, de chaque ligne de longueur $s^2 - s$ faisons émaner des faisceaux de lignes de longueur $s^3 - s^2$ pour représenter les éléments des groupes d'ordre s^3 qui appartiennent à l'exposant s^3, et ainsi de suite. Une telle figure pourrait porter le nom de *patron* (pattern, Schablone) et être désignée par la notation $P^\theta(m_\theta, m_{\theta-1}, \ldots . m_2, m_1)$ de manière à mettre en évidence les nombres dont le patron dépend. Au lieu du patron, on peut se servir du *type* qui représente la même chose sous une forme plus abrégée. Un *type* diffère d'un patron en ce que:

1). Toutes les lignes sont de longueur 1.

2). Il n'y a qu'une seule ligne* pour chaque faisceau d'émanants d'ordre s^{u+1} et d'une même portée qui émanant d'un groupe d'ordre s^u qui lui-même n'est représenté qu'implicitement comme faisant partie d'un faisceau représenté par une ligne. Un tel *type* pourra être désigné par la notation $T(\theta)$.

Cela étant ainsi, il est facile de déterminer les nombres des éléments du groupe Ξ_θ qui appartiennent à l'exposant s^θ. En effet, pour avoir le nombre des groupes monobases d'ordre s^θ, il faut multiplier le nombre des groupes monobases d'ordre s et de portée s^θ, par le nombre des émanants d'ordre s^2 et de portée s^θ qui émanant de chaque groupe monobase d'ordre s et de portée s^θ; puis multiplier le résultant par le nombre des émanants d'ordre s^3 et de portée s^θ qui émanent de tout groupe monobase d'ordre s^2 et de portée s^θ, et ainsi de suite. Cela donne pour résultat

$$\frac{s^{m_\theta}-1}{s-1} \times s^{m_{\theta-1}+m_{\theta-2}+\ldots . m_1-\theta+1}.$$

Comme chaque groupe monobase d'ordre s^θ contient $s^\theta - s^{\theta-1}$ éléments appartenant à l'exposant s^θ, le nombre de tels éléments† dans le groupe Ξ_θ est égal à

$$(s^{m_\theta} - 1)\, s^{m_{\theta-1}+m_{\theta-2}+\ldots . m_1}.$$

Comme le groupe Ξ_u, où $1 < u \leqq \theta$ est de rang u par rapport au nombre premier s et est identique à l'ensemble des solutions de l'égalité

$$x^{s^u} = 1 \qquad (\text{gr. } \Xi_u),$$

tout ce que nous avons dit du groupe Ξ_θ s'applique au groupe Ξ_u pourvu qu'on remplace le nombre θ par le nombre u. Les nombres

$$m_1,\ m_2,\ \ldots .\ m_u$$

---

* Le nombre des groupes représenté par une telle ligne peut se réduire quelquefois à zéro.

† Deux groupes monobases d'ordre s^θ ne peuvent avoir en commun un élément appartenant à l'exposant s^θ sans être identiques.

auront d'ailleurs la même valeur dans les deux cas. Le nombre des éléments du groupe Ξ_θ appartenant à l'exposant s^u est donc égal à

$$(s^{m_u}-1)\,s^{m_{u-1}+m_{u-2}+\cdots+m_1}.$$

Quant au groupe Ξ_1, il a été étudié à part dans le §7 et nous savons que le nombre de ses éléments appartenant à l'exposant s est égal à $s^{m_1}-1$. Cela étant ainsi, si l'on désigne par ξ_θ l'ordre du groupe Ξ_θ, on aura

$$\xi_\theta = 1 + (s^{m_1}-1) + \sum_{u=2}^{u=\theta}(s^{m_u}-1)\,s^{m_{u-1}+m_{u-2}+\cdots m_1} = s^{m_\theta+m_{\theta-1}+\cdots m_1}.$$

Et de même, si l'on désigne par ξ_u l'ordre du groupe Ξ_u où $0 < u \leqq \theta$, on aura

$$\xi_u = s^{m_u+m_{u-1}+\cdots+m_1},$$

9.

Passons maintenant à la détermination de l'ordre maximum d'un produit de sous-groupes monobases de Ξ_θ. Je m'occuperai d'abord du problème suivant. Étant donnés les sous-groupes monobases

$$A_1,\ A_2,\ A_3,\ \ldots.\ A_n,$$

on demande la condition nécessaire et suffisante pour qu'on puisse les combiner en un produit. Soient respectivement

$$B_1,\ B_2,\ \ldots.\ B_n$$

les émettants d'ordre s des groupes

$$A_1,\ A_2,\ \ldots.\ A_n,$$

si l'on peut combiner les groupes

$$A_1,\ A_2,\ \ldots.\ A_n,$$

il est clair qu'on pourra aussi combiner les groupes

$$B_1,\ B_2,\ \ldots.\ B_n.$$

Je dis qu'inversement, si l'on peut combiner les groupes

$$B_1,\ B_2,\ \ldots.\ B_n,$$

on pourra aussi combiner les groupes

$$A_1,\ A_2,\ \dots\dots\ A_n.$$

En effet, s'il est impossible de combiner les groupes

$$A_1,\ A_2,\ \dots\dots\ A_n$$

cela provient de ce que, par exemple, on peut combiner les groupes

$$A_1,\ A_2,\ \dots\dots\ A_k$$

où $1 \leqq k < u$, mais qu'il n'est plus possible de combiner les groupes $A_1,\ A_2, \dots\dots\ A_k$ et A_{k+1} parce qu'ils ont en commun un élément a_{k+1} différent de l'élément-unité. Soient $a_1,\ a_2,\ \dots\dots\ a_k$ les éléments des groupes

$$A_1,\ A_2,\ \dots\dots\ A_k$$

dont le produit donne a_{k+1}. Quelques-uns des éléments

$$a_1,\ a_2,\ \dots\dots\ a_k$$

peuvent être identiques à l'élément-unité, mais l'élément a_{k+1} est, d'après la supposition, différent de l'élément-unité. Cela étant ainsi, soient

$$s^{h_1},\ s^{h_2},\ \dots\dots\ s^{h_{k+1}}$$

les exposants auxquels appartiennent les éléments

$$a_1,\ a_2,\ \dots\dots\ a_{k+1}.$$

Quelques-uns des nombres $h_1,\ h_2,\ \dots\dots\ h_k$ peuvent être égaux à zéro, mais l'on aura nécessairement, d'après la supposition,

$$h_{k+1} > 0.$$

Cela étant ainsi, désignons d'une manière générale par v_u, où $u \leqq k+1$, l'unité si h_u est égal à zéro ou un, et s^{h_u-1} quand h_u est supérieur à l'unité. Soit v_s le plus grand des nombres v_u, nous aurons

$$a_1^{v_s}a_2^{v_s}\dots\dots a_k^{v_s} = a_{k+1}^{v_s}$$

et je dis que l'on aura $v_s = v_{k+1}.$

En effet, si l'on avait $v_{k+1} < v_s,$

mais $v_s = v_u$

par exemple, où $u < k + 1$, le second membre de l'égalite

$$a_1^{v_1} a_2^{v_2} \dots a_k^{v_k} = a_{k+1}^{v_{k+1}},$$

serait égal à l'unité, tandis que le premier membre serait différent de l'unité. Tout élément $a_u^{v_u}$ où $u \leqq k + 1$ est égal à l'élément-unité on appartient à l'exposant s; les éléments

$$a_1^{v_1}, a_2^{v_2}, \dots a_{k+1}^{v_{k+1}}$$

font donc partie respectivement des groupes

$$B_1, B_2, \dots B_{k+1}.$$

Il en résulte qu'on peut combiner les groupes

$$B_1, B_2, \dots B_k$$

puisqu'on peut combiner les groupes

$$A_1, A_2, \dots A_k,$$

mais qu'on ne peut plus combiner les groupes B_1, $B_2, \dots B_k$ et B_{k+1} parce qu'ils ont en commun l'élément $a_{k+1}^{v_{k+1}}$ différent de l'élément-unité. Comme cette conclusion est contraire à notre supposition, la proposition est démontrée. La condition nécessaire et suffisante pour qu'on puisse combiner les groupes

$$A_1, A_2, \dots A_n$$

en un produit, est donc qu'on puisse combiner leurs émettants d'ordre s

$$B_1, B_2, \dots B_n$$

en un produit. La valeur maximum de n est donc égale à m_1, car on peut combiner m_1 groupes monobases d'ordre s, mais il n'est plus possible d'en combiner un plus grand nombre; autrement Ξ_1 admettrait un sous-groupe d'ordre s^n supérieur à s^{m_1}. Quel est maintenant le nombre maximum de groupes monobases d'un ordre supérieur à s, qu'on puisse combiner en un produit? Les émettants d'ordre s de tels groupes seront de portée s^2 au moins. Tous ces émettants seront donc des sous-groupes de H_2 et inversement tout sous-groupe monobase d'ordre s de H_2 est de portée s^2 au moins. Le nombre maximum de sous-groupes monobases d'ordre s et de portée s^2 au moins qu'on puisse combiner en un produit, est donc égal à m_2. Le nombre maximum de groupes monobases d'ordre s^2 au moins qu'on puisse combiner en un produit est donc aussi égal à m_2. Et d'une manière générale le nombre maximum de groupes d'ordre s et de portée s^u au moins, où $u \leqq \theta$, qu'on puisse combiner en un produit, est égal à m_u. Donc le nombre maximum de groupes monobases d'ordre s^u au moins qu'on puisse combiner en un produit est aussi égal à m_u. Or on peut trouver une décomposition de Ξ_1 où il y aura m_θ groupes d'ordre s et de portée s^θ, $m_{\theta-1} - m_\theta$ groupes d'ordre s et de portée $s^{\theta-1}$, $m_{\theta-2} - m_{\theta-1}$ groupes d'ordre s et de portée $s^{\theta-2}$, enfin $m_1 - m_2$ groupes d'ordre s et de portée s.* En effet, il suffit de décomposer H_θ, puis si $H_{\theta-1}$ est différent de H_θ achever la décomposition de $H_{\theta-1}$ en commençant par la décomposition de H_θ, puis achever la décomposition de $H_{\theta-2}$ en commençant par celle de $H_{\theta-1}$ et ainsi de suite. Une telle décomposition de Ξ_1 portera le nom de *convenable* (par rapport au groupe Ξ_θ). Je me propose de rechercher le nombre des décompositions convenables de Ξ_1.

Pour obtenir une décomposition convenable de Ξ_1 il faut d'abord décomposer H_θ en facteurs monobases, ce qui se fait de

$$\frac{s^{m_\theta} - 1}{s - 1} \cdot \frac{s^{m_\theta} - s}{s - 1} \cdot \frac{s^{m_\theta} - s^2}{s - 1} \cdots\cdots \frac{s^{m_\theta} - s^{m_\theta - 1}}{s - 1} : m_\theta!$$

manières, puis achever la décomposition de $H_{\theta-1}$ ce qui se fait de

$$\frac{s^{m_{\theta-1}} - s^{m_\theta}}{s - 1} \cdot \frac{s^{m_{\theta-1}} - s^{m_\theta + 1}}{s - 1} \cdots\cdots \frac{s^{m_{\theta-1}} - s^{m_{\theta-1} - 1}}{s - 1} : (m_{\theta-1} - m_\theta)!$$

manières,† puis il faut achever la décomposition de $H_{\theta-2}$ ce qui se fait de

$$\frac{s^{m_{\theta-2}} - s^{m_{\theta-1}}}{s - 1} \cdot \frac{s^{m_{\theta-2}} - s^{m_{\theta-1} + 1}}{s - 1} \cdots\cdots \frac{s^{m_{\theta-2}} - s^{m_{\theta-2} - 1}}{s - 1} : (m_{\theta-2} - m_{\theta-1})!$$

manières, ... puis achever la décomposition de H_2 ce qui se fait de

$$\frac{s^{m_2} - s^{m_3}}{s - 1} \cdot \frac{s^{m_2} - s^{m_3 + 1}}{s - 1} \cdots\cdots \frac{s^{m_2} - s^{m_2 - 1}}{s - 1} : (m_2 - m_3)!$$

manières, enfin achever la décomposition de $H_1 = \Xi_1$ ce qui se fait de

$$\frac{s^{m_1} - s^{m_2}}{s - 1} \cdot \frac{s^{m_1} - s^{m_2 + 1}}{s - 1} \cdots\cdots \frac{s^{m_1} - s^{m_1 - 1}}{s - 1} : (m_1 - m_2)!$$

manières. Le produit de ces nombres donne le nombre des décompositions convenables de Ξ_1. Si dans une décomposition convenable de Ξ_1 on remplace chaque groupe d'ordre s par sont émanant de l'ordre le plus élevé, le produit des groupes monobases qu'on obtient de cette manière sera d'ordre

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* Quelques-unes des différences $m_{u-1} - m_u$ peuvent se réduire à zéro.

† Quand on a $H_{\theta-1} = H_\theta$ il faut remplacer l'expression précédente par l'unité et de même pour les autres cas où l'on aurait $H_u = H_{u+1}$.

$$s^{\theta m_\theta+(\theta-1)(m_{\theta-1}-m_\theta)+(\theta-2)(m_{\theta-2}-m_{\theta-1})+\dots 2(m_2-m_3)+(m_1-m_2)}$$
$$=s^{m_\theta+m_{\theta-1}+\dots m_2+m_1}=\xi_\theta,$$

de sorte que nous avons obtenu ainsi une décomposition de Ξ_θ en un produit de groupes monobases. Je dis que cette décomposition est univoque quant au nombre de ses groupes monobases composants d'un ordre déterminé.

En effet, si le groupe Ξ_θ est décomposable en un produit contenant n_θ groupes monobases d'ordre s^θ, $n_{\theta-1}-n_\theta$ groupes monobases d'ordre $s^{\theta-1}$, ... n_u-n_{u+1} groupes monobases d'ordre s^u, ... n_1-n_2 groupes monobases d'ordre s, on aura

$$\xi_\theta=s^{m_\theta+m_{\theta-1}+\dots m_2+m_1}=s^{n_\theta+n_{\theta-1}+\dots n_2+n_1},$$

d'où $$m_\theta+m_{\theta-1}+\dots m_2+m_1=n_\theta+n_{\theta-1}+\dots n_2+n_1.$$

Or le nombre maximum de groupes monobases d'ordre s^u au moins, qu'on puisse combiner en un produit étant égal à m_u, on aura pour toute valeur de u non supérieure à θ

$$n_u\leqq m_u$$

et par suite en vertu de l'égalité

$$n_\theta+n_{\theta-1}+\dots n_1=m_\theta+m_{\theta-1}+\dots m_1,$$
$$n_u=m_u.$$

Cherchons maintenant le nombre des décompositions de Ξ_θ qui correspondent à une décomposition convenable de Ξ_1. Nous avons vu que chaque groupe d'ordre s et de portée s^u admet

$$s^{m_{u-1}+m_{u-2}+\dots m_2+m_1-u+1}$$

émanants d'ordre s^u. Le nombre des décompositions de Ξ_θ qui correspondent à une décomposition convenable de Ξ_1 est donc égal à

$$s^{m_\theta(m_{\theta-1}+m_{\theta-2}+\dots m_1-\theta+1)+(m_{\theta-1}-m_\theta)(m_{\theta-2}+m_{\theta-3}+\dots m_1-\theta+2)+\dots(m_2-m_3)(m_1-1)}$$
$$=s^{m_\theta m_{\theta-1}+m_{\theta-1}m_{\theta-2}+\dots m_3m_2+m_2m_1-(m_\theta+m_{\theta-1}+\dots m_1)}.$$

En multipliant ce nombre par le nombre des décompositions convenables de Ξ_1, on aura le nombre des décompositions de Ξ_θ en un produit de groupes monobases.

Nous avons vu dans le §6 que tout groupe monobase dont l'ordre est une puissance de nombre premier, est simple. Il s'ensuit que tout sous-groupe monobase de Ξ_θ est simple. Inversement, tout sous-groupe simple de Ξ_θ est monobase. En effet, soit A un tel sous-groupe et w son rang par rapport au nombre premier s, le groupe A sera identique à l'ensemble des solutions de l'égalité

$$x^{s^w} = 1 \qquad (\text{gr. } A).$$

Le groupe A est donc égal à un groupe monobase ou à un produit de groupes monobases. La dernière supposition étant à excluse, le groupe A sera nécessairement monobase.

J'aurai besoin encore du lemme suivant.

Soit A' un groupe monobase d'ordre s et de portée s^u, on peut faire figurer explicitement, dans une décomposition de Ξ_θ, le groupe A' comme émettant d'un groupe monobase composant d'ordre s^u. En effet soit d'abord $u < \theta$, le groupe A' étant de portée s^u, le groupe H_u renfermera des éléments ne faisant pas partie de H_{u+1}. Décomposons d'abord H_{u+1}, mais de manière que cette décomposition de H_{u+1} puisse servir de commencement à une décomposition convenable de Ξ_1 et puis achevons la décomposition de H_u en commençant par A' et après cela achevons la décomposition convenable de Ξ_1 d'une manière quelconque. Toute décomposition de Ξ_θ correspondant à cette décomposition convenable de Ξ_1 aura les conditions requises. Pour $u = \theta$, on ferait commencer la décomposition convenable de Ξ_1 par le groupe A'.

10.

Nous avons vu que le groupe Ξ_θ peut être décomposé en un produit de groupes monobases*

$$\Xi_\theta = A_1^{s^\theta} A_2^{s^\theta} \dots\dots A_{m_\theta}^{s^\theta} A_{m_{\theta+1}}^{s^{\theta-1}} A_{m_{\theta+2}}^{x^{\theta-1}} \dots\dots A_{m_{\theta-1}}^{s^{\theta-1}} A_{m_{\theta-1}+1}^{s^{\theta-2}} \dots\dots A_{m_2}^{s^2} A_{m_2+1}^{s} A_{m_2+2}^{s} \dots\dots A_{m_1}^{s},$$

et que cette décomposition est univoque en ce sens qu'elle renferme un nombre déterminé de groupes d'un même ordre.

Soient $$a_1, a_2, \dots\dots a_{m_1}$$

les bases des groupes monobases composants d'une décomposition de Ξ_θ et

$$v_1, v_2, \dots\dots v_{m_1}$$

---

* Le nombre m_u peut être égal à m_{u+1}; alors les groupes d'ordre s^u disparaissent dans la décomposition.

es exposants auxquels ces bases appartiennent, l'expression

$$a_1^{x_1}a_2^{x_2}\ldots\ldots a_{m_1}^{x_{m_1}} \qquad (\text{où } x_k = 0,\ 1,\ 2,\ \ldots\ldots x_{m_k-1})$$

représentera tous les éléments du groupe Ξ_θ et chacun d'une seule manière. Je dis que si

$$b_1,\ b_2,\ \ldots\ldots b_n$$

sont des éléments du groupe Ξ_θ et que l'expression telle que

$$b_1^{y_1}b_2^{y_2}\ldots\ldots b_n^{y_n},$$

où

$$y_1,\ y_2,\ \ldots\ldots y_n$$

sont des nombres entiers quelconques, est susceptible de représenter tous les éléments du groupe Ξ_θ, on a

$$n \geqq m_1.$$

En effet, soient $s^{w_1},\ s^{w_2},\ \ldots\ldots s^{w_n}$

les exposants auxquels appartiennent les éléments

$$b_1,\ b_2,\ \ldots\ldots b_n,$$

il suffit de faire parcourir à y_k toutes les valeurs depuis 0 jusqu'à $s^{w_k} - 1$; si l'on faisait aller y_k au delà de $s^{w_k} - 1$, on n'obtiendrait aucune valeur nouvelle.

Soient $B_1,\ B_2,\ B_3,\ \ldots\ldots B_n$

les groupes auxquels $b_1,\ b_2,\ \ldots\ldots b_n$

servent de bases et $C_1,\ C_2,\ \ldots\ldots C_n$

leurs émettants d'ordre s. Si l'on peut combiner les groupes

$$B_1,\ B_2,\ \ldots\ldots B_n,$$

on aura évidemment

$$\Xi_\theta = B_1 B_2 \ldots\ldots B_n$$

et $n = m_1.$

Si non, considérons les émettants

$$C_1,\ C_2,\ \ldots\ldots C_n;$$

s'il y en a des égaux entre eux, C_k et C_l par exemple, on peut remplacer les groupes B_k et B_l par d'autres et il y aura deux cas à considérer :

1). Un des deux groupes B_k et B_l est un sous-groupe de l'autre, B_l est un sous-groupe de B_k par exemple; dans ce cas on peut remplacer B_k et B_l par le seul groupe B_k; tout élément qui est susceptible d'être représenté comme un produit d'un élément de B_k par un élément de B_l fait partie de B_k.

2). Aucun des deux groupes B_k et B_l n'est sous-groupe de l'autre. Dans ce dernier cas on peut remplacer les groupes B_k et B_l par deux autres groupes monobases B_k' et B_l' ayant des émettants C_k' et C_l' d'ordre s différents entre eux et de manière que tout élément qui est susceptible d'être représenté comme un produit d'un élément du groupe B_k par un élément du groupe B_l, soit aussi représentable comme un produit d'un élément du groupe B_k' par un élément du groupe B_l'. En effet, les émettants d'ordre s des groupes B_k et B_l étant égaux entre eux, soit d'une manière générale $C_k^{s^u}$ l'émettant de B_k d'ordre s^u et de même $C_l^{s^u}$ l'émettant de B_l d'ordre s^u; on aura

$$C_k^{s^u} = C_l^{s^u}$$

jusqu'à une certaine valeur de u égale à v par exemple où

$$1 \leqq v,$$
$$v < w_k$$

et

$$v < w_l.$$

On peut supposer d'ailleurs

$$w_k \geqq w_l.$$

Cela étant ainsi, les élément $b_k^{s^{w_k - v}}$ et $b_l^{s^{w_l - v}}$ appartiendront à l'exposant s^v et feront partie de $C^{s^v} = C_k^{s^v} = C_l^{s^v}$ par conséquent; on aura donc

$$b_l^{s^{w_l - v}} = b_k^{h s^{w_k - v}} \qquad (\text{gr. } \Xi_\theta),$$

où h n'est pas divisible par s. Posons

$$b_k^{h s^{w_k - v}} b_l^{-1} = L \qquad (\text{gr. } \Xi_\theta),$$

je dis que L appartiendra à l'exposant $s^{w_l - v}$. En effet, on a

$$L^{s^{w_l - v}} = b_k^{h s^{w_k - v}} b_l^{- s^{w_l - v}} = 1 \qquad (\text{gr. } \Xi_\theta),$$

tandis que si l'on avait

$$L^{s^{w_l - v - 1}} = b_k^{h s^{w_k - v - 1}} b_l^{- s^{w_l - v - 1}} = 1 \qquad (\text{gr. } \Xi_\theta)$$

on en tirerait

$$b_k^{h s^{w_k - v - 1}} = b_l^{s^{w_l - v - 1}} \qquad (\text{gr. } \Xi_\theta).$$

Les groupes B_k et B_l auraient donc en commun un élément appartenant à l'exposant s^{v+1} et par suite aussi l'émettant d'ordre s^{v+1} contrairement à la supposition. Désignons par A le groupe qui a L pour base. Je dis que le groupe B_k n'a de commun avec A que l'élément-unité. En effet, si ces groupes avaient en commun en élément appartenant à l'exposant $s^{w_l - v}$, le groupe A serait un sous-groupe de B_k et

$$b_l = L^{-1} b_k^{hs^{w_k} - w_l}$$

ferait partie du groupe B_k; le groupe B_l serait donc un sous-groupe de B_k contrairement à la supposition. De même, si B_k et A avaient en commun un élément L^{zs^t} où $0 < t < w_l - v$ et z non divisible par s, on aurait

$$b_k^{hzs^{w_k} - w_l + t} b_l^{-zs^t} = L^{zs^t} \qquad (\text{gr. } \Xi_\theta)$$

et les groupes B_k et B_l auraient en commun un élément $b_l^{-zs^t}$ appartenant à un exposant $s^{w_l - t}$ où $w_l - t > v$, et par suite les groupes B_k et B_l auraient aussi en commun l'émettant d'ordre $s^{w_l - t}$ contrairement à la supposition. Soit donc

$$B_k A = \psi \qquad (\text{gr. } \Xi_\theta).$$

Je dis que le groupe ψ d'ordre $s^{w_k + w_l - v}$ contient tous les éléments qu'on peut obtenir en combinant un élément du groupe B_k avec un élément du groupe B_l. En effet, un tel élément

$$b_k^f b_l^g$$

sera aussi égal à

$$b_k^f L^{-g} b_k^{ghs^{w_k} - w_l} = b_k^{f + ghs^{w_k} - w_l} L^{-g}$$

et ferait partie de ψ par conséquent. Si dans la suite

$$B_1, B_2, \ldots . B_n$$

on remplace B_l par A, tout élément de Ξ_θ pourra encore être représenté comme un produit d'éléments de la nouvelle suite de groupes que nous venons d'obtenir. Cette nouvelle suite diffère de la précédente en ce qu'un groupe A y est d'un ordre $s^{w_l - v}$ inférieur à l'ordre s^{w_l} du groupe correspondant de la première suite. Tous les autres groupes correspondants des deux suites sont de même ordre.

Si la nouvelle suite renferme des groupes ayant des émettants d'ordre s identiques, on pourra appliquer la même transformation et l'on remplacera encore un des groupes par un autre d'un ordre inférieur. Comme cela ne peut aller à l'infini, on finira par arriver à une suite que nous désignerons encore par

$$B_1, B_2, \ldots . B_n$$

et dont les émettants d'ordre s

$$C_1, C_2, \ldots . C_n$$

seront tous différents entre eux. Le nombre n a d'ailleurs une valeur égale ou inférieure à celle qu'il avait précedemment. Cela étant ainsi, si l'on peut combiner les émettants

$$C_1,\ C_2,\ \ldots.\ C_n,$$

on aura

$$\Xi_\theta = B_1 B_2 \ldots. B_n$$

et $n = m_1$ par conséquent. Si non, supposons qu'on puisse combiner les émettants

$$C_1,\ C_2,\ \ldots.\ C_k,$$

mais que C_{k+1} ait en commun avec le groupe

$$C = C_1 C_2 \ldots. C_k$$

un élément différent de l'élément-unité; C_{k+1} sera alors un sous-groupe de C. Posons de même

$$B = B_1 B_2 \ldots. B_k$$

et remplaçons la décomposition précédente par une autre

$$B = B_1' B_2' \ldots. B_k',$$

où l'émettant C_{k+1} entre explicitement comme émettant C_h' d'un groupe $B'h$ par exemple, où $h \leqq k$. Si dans la dernière suite, on remplace les groupes

$$B_1,\ B_2,\ \ldots.\ B_k$$

par les groupes

$$B_1',\ B_2',\ \ldots.\ B_k'$$

on obtient une nouvelle suite dont les émettants d'ordre s ne seront plus différents entre eux. Les ordres des groupes correspondants seront d'ailleurs les mêmes dans les deux suites. Il y aura donc lieu d'appliquer une nouvelle transformation à la suite

$$B_1',\ B_2',\ \ldots.\ B_k',\ B_{k+1},\ \ldots.\ B_n$$

et de continuer à appliquer une .telle transformation jusqu'à ce qu'on obtienne une suite que je désignerai encore par

$$B_1,\ B_2,\ \ldots.\ B_n$$

et qui n'aura plus d'émettants égaux. Si l'on ne peut combiner les émettants d'ordre s de cette nouvelle suite, on la transformera encore en une suite ayant des émettants égaux.

On voit qu'à toute suite telle que

$$B_1,\ B_2,\ \ldots.\ B_n$$

dont les groupes ne sont pas susceptible d'être combinés en un produit, on peut appliquer *soit* une opération ϕ qui consiste à supprimer un groupe tel que B_k ou à le remplacer par un autre d'un ordre inférieur et cela de manière que la nouvelle suite puisse représenter, par la composition, les mêmes éléments que celle dont elle provient; *soit*, dans le cas où l'opération ϕ n'est pas applicable, une opération ψ qui consiste à remplacer quelques uns des groupes de la suite par d'autres de même ordre et cela de manière que la nouvelle suite puisse représenter, par la composition, les mêmes éléments que celle dont elle provient et que la nouvelle suite soit susceptible d'être l'objet de l'opération ϕ.

De telles opérations doivent nécessairement aboutir, et l'on finira par arriver à une suite de groupes

$$B_1, B_2, \ldots . B_n$$

susceptible d'être combinés en un produit et de représenter, par la composition, tous les éléments du groupe Ξ_0. On aura donc

$$\Xi_0 = B_1 B_2 \ldots . B_n$$

et par suite

$$n = m_1.$$

La dernière valeur de n est, comme on le voit égale à m_1; il s'ensuit que la première valeur de n est égale ou supérieure à m_1. C'est précisement ce que nous voulions établir.

(*à suivre.*)

10 juillet 1888.

On the Theory of Groups.

BY PROF. CAYLEY.

I refer to my papers on the theory of groups as depending on the symbolic equation $\theta^n = 1$, Phil. Mag., vol. VII (1854), pp. 40–47 and 408–409; also vol. XVIII (1859), pp. 34–37; and "On the Theory of Groups," *Amer. Journ. of Math.*, vol. I (1878), pp. 50–52, and "The Theory of Groups: Graphical Representation," id., pp. 174–176; also to Mr. Kempe's "Memoir on the Theory of Mathematical Form," Phil. Trans., vol. 177 (1886), pp. 1–70, see the section "Groups containing from one to twelve units," pp. 37–43, with the diagrams given therein. Mr. Kempe's paper has recalled my attention to the method of graphical representation explained in the second of the two papers of 1878, and has led me to consider, in place of a diagram as there given for the independent substitutions, a diagram such as those of his paper, for all the substitutions. I call this a colourgroup; viz. for the representation of a substitution-group of s substitutions upon the same number of letters, or say of the order s, we employ a figure of s points (in space or in a plane) connected together by coloured lines, and called a colourgroup.

I remark that up to $s = 11$, the first case of any difficulty is that of $s = 8$, and that the 5 groups of this order were determined in my papers of 1854 and 1859. For the order 12, Mr. Kempe has five groups, but one of these is non-existent, and there is a group omitted; the number is thus $= 5$.

The colourgroup consists of s points joined in pairs by $\frac{1}{2}s(s-1)$ coloured lines under prescribed conditions. A line joining two points is in general regarded as a vector drawn *from* one *to* the other of the two points; the currency is shown by an arrow, and in speaking of a line ab we mean the line from a to b. But we may have a line regarded as a double line, drawn from each to the other of the two points; the arrow is then omitted, and in speaking of such a line ab we mean the line from b to a and from a to b. A fresh condition is that for a given colour there shall be one and only one line *from* each of the points, and one and only one line *to* each of the points. We may have through two points a, b only the line ab of the given colour; this is then a double line regarded as drawn from a to b and from b to a; and there is thus one and only one line of the colour from each of these points and to each of these points. The condition implies that the lines of a given colour form either a single polygon or a set of polygons, with a continuous currency round each polygon; for instance, there may be a pentagon $abcde$, meaning thereby the pentagon formed by the lines drawn from a to b, from b to c, from c to d, from d to e, and from e to a. An arrow on one of the sides is sufficient to indicate the currency. In the case of a double line we have a polygon of two points, or say a digon.

There is a further condition which, after the necessary explanation of the meaning of the terms, may be concisely expressed as follows: Each route must be of independent effect, and (as will readily be seen) this implies that the lines of a given color must form either a single polygon or else two or more polygons each of the same number of points: thus if $s = ks_1$, they may form k s_1-gons; in particular, if s be even, they may form $\frac{1}{2}s$ digons.

To explain the foregoing statement, first as to the term "route." I denote the several colours by capital letters, $R =$ red, $G =$ green, $B =$ blue, etc. Any capital or combination of capitals determines a route; R means go along a red line; $RRBG$, go along a red line, a red line, a blue line, a green line, and so in other cases. Given the starting point, or initial, the route determines the several points passed through, and the point arrived at, or terminal, thus $aRRBG = abefk, = k$, means that the route $RRBG$ leads from a through b, e, f to k, viz. that the red line from a leads to b, the red line from b leads to e, the blue line from e leads to f, and the green line from f leads to k. We may give in this way the Itinerary, or write simply $aRRBG = k$, meaning that the route leads from a to k. We may of course write R^2 for RR, and so in other cases. A single capital, as already mentioned, is a route, but it may for distinction be called a stage. A stage, and thence also a route, may be *reversed;* R^{-1} means go along the red line drawn to the point; if $aR = b$, then $bR^{-1} = a$; and so if $aRRBG = abefk, = k$, then $kG^{-1}B^{-1}R^{-1}R^{-1} = kfeba, = a$; $R^{-1}R^{-1} = R^{-2}$, and so in other cases.

The effect of a route depends in general on the initial point: thus, a route may lead from a point a to itself, or say it may be a circuit from a; and it may not be a circuit from another point b. And similarly two different routes each leading from a point a, to one and the same point x, or say two routes equivalent for the initial point a, may not be equivalent for a different initial point b. Thus we cannot in general say simpliciter that a route is a circuit, or that two different routes are equivalent. But the figure may be such as to render either of these locutions, and if either, then each of them, admissible. For it is easy to see that if every route which is a circuit from any one initial point is also a circuit from every other initial point, then two routes which are equivalent for any one initial point will be equivalent for every other initial point. And conversely, if in every case where two different routes are equivalent for any one initial point, they are equivalent for every other initial point, then every route which is a circuit from any one initial point is a circuit from every other initial point; and we express this by saying that every route is of independent effect: this explains the meaning of the foregoing statement of the condition which is to be satisfied by a colourgroup.

It is at once evident that a colourgroup, quâ figure where each route is of independent effect, furnishes a graphical representation of the substitution-group and gives the square by which we define such group. For in the colourgroup of s points we have the route from a point to itself and the routes to each of the other $(s-1)$ points, in all s non-equivalent routes; and if starting from a given arrangement, say $abcd\ldots.$, of the s points, we go by one of these routes from the several points $a, b, c, d, \ldots.$ successively, we obtain a different arrangement of these points. Observe that this is so; the same point cannot occur twice, for if it did, there would be a route leading from two different points b, f to one and the same point x, or the reverse route from x would lead to two different points b, f. The route from a point to itself which leaves each point unaltered, and thus gives the primitive arrangement $abcd\ldots.$, may be called the route 1. Taking this route and the other $(s-1)$ routes successively, we obtain s different arrangements of the points, or say a square, each line of which is a different arrangement of the points. And not only are the arrangements different, but we cannot have the same point twice in any column, for this would mean that there were two different routes leading from a point to one and the same point x; hence each column of the square will be an arrangement of the s points. We have thus the substitution-group of the s points or letters; the s routes, or say the route 1 and the other $(s-1)$ routes, are the substitutions of the group.

The complete figure is called the colourgroup. As already mentioned, the lines of any colour form either a single polygon or two or more polygons each of the same number of points. The number of lines of a given color is thus $= s$, or when the polygons are digons (which implies s even), the number is $= \frac{1}{2} s$. The number of colours is thus $= \frac{1}{2}(s-1)$ at least, and $= (s-1)$ at most. A general description of the figure may be given as in the annexed Table. Thus for the group $6B$ we have $\begin{array}{r} R.\ 2\ \text{3gons} = 6 \\ B, G, Y.\ (3\ \text{2gons})^3 = 9 \\ \hline \underline{\underline{15}} \end{array}$; we have the red lines forming two trigons, 6 lines, and the blue, green and yellow lines each forming three digons, together $3 \times 3, = 9$ lines, in all $15, = \frac{1}{2}\, 6.5$ lines. Such description, however, does not indicate the currencies, and it is thus insufficient for the determination of the figure. But the figure is completely determined by means of the substitutions as given in the outside column of the square, thus $R = (abc)(dfe)$ shows that the red lines form the two triangles *abc*, *dfe* with these currencies, $G = (ad)(be)(cf)$, that the green lines form the three digons *ad*, *be*, *cf*, and so for the other two colours B and Y.

The lines of a colour may be spoken of as a colour, and the lines of a colour or of two or more colours as a colourset. The colourset either does not connect together all the points, and it is then a broken set; or it does connect together all the points, and it is then a bondset. A bondset not containing any superfluous colour is termed a bond, viz. a bond is a colourset which connects together all the points, but which is moreover such that if any one of the colours be omitted it becomes a broken set. The word colour is used as a prefix, colourset as above, colourbond, etc., and so also with a numeral, a twocolourbond is a bond with two colours, and so in other cases. Observe that we may very well have for instance a threecolourbond, and also a twocolour or a onecolourbond, only the colours or colour hereof must not be included among those of the threecolourbond, for this would then contain a superfluous colour or colours and would not be a bond.

A colourgroup may contain a onecolourbond, viz. this is the case when all the points form a single polygon; it is then said to be unibasic. If it contains no onecolourbond but contains a twocolourbond, it is bibasic; if it contains no onecolourbond or twocolourbond but contains a threecolourbond, it is tribasic, and so on. In all cases the number of bonds (onecolour-, twocolour-, etc.) may very well be and in general is greater than one; thus a unibasic colourgroup will in general contain several onecolourbonds, a bibasic colourgroup several twocolourbonds, and so on.

The bond of the proper number of colours completely determines the colourgroup; in fact the colourbond gives the route from any one point to each of the other $(s-1)$ points; that is, it determines all the s routes, and consequently the colourgroup. The only type of onecolourbond is the polygon of the s points; we have thus for any value whatever of s a unibasic colourgroup which may be called sA. The theory is well known. If s be a prime number, the number of colours is $= \frac{1}{2}(s-1)$, each colour gives a polygon through the s points, so that we have here only onecolourbonds; but in other cases we have broken sets, and there will be in general (but not for all such values of s) twocolourbonds. Observe, moreover, that for s a prime number the only colourgroup is the foregoing unibasic group sA. I have just employed, and shall again do so, the word type; the sense in which it is used does not, I think, require explanation.

Passing next to the bibasic colourgroups sB: there will be in general for a given composite value of s several of these, and in the absence of a more complete classification they may be called $sB1$, $sB2$, etc. In regard hereto observe that supposing for a given value of s that we know all the different types of twocolourbond, each one of these gives rise to a group, but this is not in every case a group sB; any twocolourbond contained in the corresponding group sA would give rise to the group sA which contained it, and not to a group sB. We have thus in the first instance to reject those twocolourbonds which are contained in the group sA. But attending only to the remaining twocolourbonds, these give rise each of them to a group sB, but the groups thus obtained are not in every case distinct groups. For looking at the converse question, suppose that for a given value of s we know the group sA and also the several groups sB. In any one of these groups, combining in pairs the several colours hereof RG, RY, GY, etc., we ascertain how many of these combinations are distinct types of twocolourbond, and in this manner reproduce the whole series of types of twocolourbond, not in general singly, but in sets, those which arise from sA, those which arise from $sB1$, those which arise from $sB2$, etc.; and we thus have (it may be) several types of twocolourbond each leading to the unibasic group

sA, several types each leading to the bibasic group $sB1$, several each leading to $sB2$, and so on.

The like considerations would apply to the tribasic colourgroups sC. Supposing that we had for a given value of s the several distinct types of three-colourbond, it would be necessary first to exclude from consideration those which give rise to a unibasic group sA or a bibasic group sB, and then to consider what sets out of the remaining types give rise to distinct tribasic groups sC. But in the table we have only one case $8C$ of a tribasic group.

I give now a table of the several groups $s = 2$ to 12, viz. these are as above: A, unibasic; B, bibasic; C, tribasic; the several groups being

$2A$,	$3A$,	$4A$,	$5A$,	$6A$,	$7A$,	$8A$,	$9A$,	$10A$,	$11A$,	$12A$,
		$4B$,		$6B$,		$8B1$,	$9B$,	$10B$,		$12B1$,
						$8B2$,				$12B2$,
						$8B3$,				$12B3$,
						$8C$,				$12B4$,

in all 23 groups.

TABLE OF THE GROUPS 2 TO 12.

$2A$

a	*b*
b	*a*

1 colour.

$1 \quad = 1$

$R = (ab) = R$

R. 1 digon $\underline{1}$

$\underline{\underline{1}}$

$3A$

a	*b*	*c*
b	*c*	*a*
c	*a*	*b*

1 colour.

$1 \quad = 1 \quad = 1$

$R \ = (abc) = R$

$R^2 = (acb) = R^{-1}$

R. 1 3gon $\underline{3}$

$\underline{\underline{3}}$

$4A$

a	*b*	*c*	*d*
b	*c*	*d*	*a*
c	*d*	*a*	*b*
d	*a*	*b*	*c*

2 colours.

$1 \quad = 1 \qquad = 1$

$R \ = (abcd) \quad = R$

$R^2 = (ac)(bd) = G$

$R^3 = (adcb) \quad = R^{-1}$

R. 1 4gon 4

G. 2 digons $\underline{2}$

$\underline{\underline{6}}$

4B

a	*b*	*c*	*d*
b	*a*	*d*	*c*
c	*d*	*a*	*b*
d	*c*	*b*	*a*

3 colours.

$1 \ = 1 \quad = 1$

$R \ = (ab)(cd) = R$

$G \ = (ac)(bd) = G$

$RG = (ad)(bc) = Y$

$R, G, Y.$ (2 digons)^3 $\underline{6}$

$\underline{\underline{6}}$

5A

a	*b*	*c*	*d*	*e*
b	*c*	*d*	*e*	*a*
c	*d*	*e*	*a*	*b*
d	*e*	*a*	*b*	*c*
e	*a*	*b*	*c*	*d*

2 colours.

$1 \ = 1 \quad = 1$

$R \ = (abcde) = R$

$R^2 = (acebd) = G$

$R^3 = (adbec) = G^{-1}$

$R^4 = (aedcb) = R^{-1}$

$R, G.$ (1 5gon)^2 $\underline{10}$

$\underline{\underline{10}}$

6A

a	*b*	*c*	*d*	*e*	*f*
b	*c*	*d*	*e*	*f*	*a*
c	*d*	*e*	*f*	*a*	*b*
d	*e*	*f*	*a*	*b*	*c*
e	*f*	*a*	*b*	*c*	*d*
f	*a*	*b*	*c*	*d*	*e*

3 colours.

$1 \ = 1 \quad = 1$

$R \ = (abcdef) \quad = R$

$R^2 = (ace)(bdf) \quad = G$

$R^3 = (ad)(be)(cf) = Y$

$R^4 = (aec)(bfd) \quad = G^{-1}$

$R^5 = (afedcb) \quad = Y^{-1}$

R.	1 6gon	6
G.	2 3gons	6
Y.	3 digons	$\underline{3}$
		$\underline{\underline{15}}$

6B

a	*b*	*c*	*d*	*e*	*f*
b	*c*	*a*	*f*	*d*	*e*
c	*a*	*b*	*e*	*f*	*d*
d	*e*	*f*	*a*	*b*	*c*
e	*f*	*d*	*c*	*a*	*b*
f	*d*	*e*	*b*	*c*	*a*

4 colours.

$1 \quad = 1 \quad = 1$

$R \quad = (abc)(dfe) \ = R$

$R^2 \quad = (acb)(def) \ = R^{-1}$

$G \quad = (ad)(be)(cf) = G$

$RG \ = (ae)(bf)(cd) \ = Y$

$R^2G = (af)(bd)(ce) = B$

R.	2 3gons	6
G, Y, B.	(3 digons)^3	$\underline{9}$
		$\underline{\underline{15}}$

7*A*

a	*b*	*c*	*d*	*e*	*f*	*g*
b	*c*	*d*	*e*	*f*	*g*	*a*
c	*d*	*e*	*f*	*g*	*a*	*b*
d	*e*	*f*	*g*	*a*	*b*	*c*
e	*f*	*g*	*a*	*b*	*c*	*d*
f	*g*	*a*	*b*	*c*	*d*	*e*
g	*a*	*b*	*c*	*d*	*e*	*f*

3 colours.

$1 \;= 1 \qquad = 1$

$R \;= (abcdefg) = R$

$R^2 = (acegbdf) = G$

$R^3 = (adgcfbe) = Y$

$R^4 = (aebfcgd) = Y^{-1}$

$R^5 = (afdbgec) = G^{-1}$

$R^6 = (agfedcb) = R^{-1}$

R, *G*, *Y*. (1 7gon)^3 21

21

8*A*

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*
b	*c*	*d*	*e*	*f*	*g*	*h*	*a*
c	*d*	*e*	*f*	*g*	*h*	*a*	*b*
d	*e*	*f*	*g*	*h*	*a*	*b*	*c*
e	*f*	*g*	*h*	*a*	*b*	*c*	*d*
f	*g*	*h*	*a*	*b*	*c*	*d*	*e*
g	*h*	*a*	*b*	*c*	*d*	*e*	*f*
h	*a*	*b*	*c*	*d*	*e*	*f*	*g*

4 colours.

$1 \;= 1 \qquad = 1$

$R \;= (abcdefgh) \qquad = R$

$R^2 = (aceg)(bdef) \qquad = Y$

$R^3 = (adgbehcf) \qquad = G$

$R^4 = (ae)(bf)(cg)(dh) = B$

$R^5 = (afchebgd) \qquad = G^{-1}$

$R^6 = (agec)(bhfd) \qquad = Y^{-1}$

$R^7 = (ahgfedcb) \qquad = R^{-1}$

R, *G*.	(1 8gon)^2	16
Y.	2 4gons	8
B.	4 digons	4
		28

8B1

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*
b	*c*	*d*	*a*	*f*	*g*	*h*	*e*
c	*d*	*a*	*b*	*g*	*h*	*e*	*f*
d	*a*	*b*	*c*	*h*	*e*	*f*	*g*
e	*f*	*g*	*h*	*a*	*b*	*c*	*d*
f	*g*	*h*	*e*	*b*	*c*	*d*	*a*
g	*h*	*e*	*f*	*c*	*d*	*a*	*b*
h	*e*	*f*	*g*	*d*	*a*	*b*	*c*

5 colours.

$1 \quad = 1 \qquad = 1$

$R \quad = (abcd)(efgh) \qquad = R$

$R^2 \quad = (ac)(bd)(eg)(fh) = Y$

$R^3 \quad = (adcb)(ehgf) \qquad = R^{-1}$

$G \quad = (ae)(bf)(cg)(dh) = G$

$RG \quad = (afch)(bgde) \qquad = I$

$R^2G = (ag)(bh)(ce)(df) \quad = B$

$R^3G = (ahcf)(bedg) \qquad = I^{-1}$

R, *I*.	$(2\ 4\text{gons})^2$	16
Y, *G*, *B*.	$(4\ \text{digons})^3$	12
		28

8B2

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*
b	*c*	*d*	*a*	*h*	*e*	*f*	*g*
c	*d*	*a*	*b*	*g*	*h*	*e*	*f*
d	*a*	*b*	*c*	*f*	*g*	*h*	*e*
e	*f*	*g*	*h*	*a*	*b*	*c*	*d*
f	*g*	*h*	*e*	*d*	*a*	*b*	*c*
g	*h*	*e*	*f*	*c*	*d*	*a*	*b*
h	*e*	*f*	*g*	*b*	*c*	*d*	*a*

6 colours.

$1 \quad = \qquad = 1$

$R \quad = (abcd)(ehgf) \qquad = R$

$R^2 \quad = (ac)(bd)(eg)(fh) = Y$

$R^3 \quad = (adcb)(efgh) \qquad = R^{-1}$

$G \quad = (ae)(bf)(cg)(dh) = G$

$RG \quad = (af)(bg)(ch)(de) = I$

$R^2G = (ag)(bh)(ce)(df) \quad = B$

$R^3G = (ah)(be)(cf)(dg) = O$

R.	2 4gons	8
Y, *G*, *I*, *B*, *O*.	$(4\ \text{digons})^5$	20
		28

8*B*3

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*
b	*c*	*d*	*a*	*h*	*e*	*f*	*g*
c	*d*	*a*	*b*	*g*	*h*	*e*	*f*
d	*a*	*b*	*c*	*f*	*g*	*h*	*e*
e	*f*	*g*	*h*	*a*	*b*	*c*	*d*
f	*g*	*h*	*e*	*d*	*a*	*b*	*c*
g	*h*	*e*	*f*	*c*	*d*	*a*	*b*
h	*e*	*f*	*g*	*b*	*c*	*d*	*a*

4 colours.

$1 \quad = 1$

$R = (abcd)(efgh) = R$

$R^2 = (ac)(bd)(ef)(gh) = Y$

$R^3 = (adcb)(ehgf) = R^{-1}$

$G = (aecg)(bhdf) = G$

$R^3G = (afch)(bedg) = B$

$R^2G = (agce)(bfdh) = G^{-1}$

$RG = (ahcf)(bgde) = B^{-1}$

R, *G*, *B*. **(2 4gons)^3** **24**

Y. **4 digons** **4**

28

8*C*

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*
b	*a*	*d*	*c*	*f*	*e*	*h*	*g*
c	*d*	*a*	*b*	*g*	*h*	*e*	*f*
d	*c*	*b*	*a*	*h*	*g*	*f*	*e*
e	*f*	*g*	*h*	*a*	*b*	*c*	*d*
f	*e*	*h*	*g*	*b*	*a*	*d*	*c*
g	*h*	*e*	*f*	*c*	*d*	*a*	*b*
h	*g*	*f*	*e*	*d*	*c*	*b*	*a*

7 colours.

$1 = 1 = 1$

$R = (ab)(cd)(ef)(gh) = R$

$G = (ac)(bd)(eg)(fh) = G$

$RG = (ad)(bc)(eh)(fg) = B$

$Y = (ae)(bf)(cg)(dh) = Y$

$RY = (af)(be)(ch)(dg) = I$

$GY = (ag)(bh)(ce)(df) = O$

$RGY = (ah)(bg)(cf)(de) = V$

R, *G*, *B*, *Y*, *I*, *O*, *V*. **(4 digons)^7** **28**

28

9A

										4 colours.
a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	1	$= 1$
b	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*a*	$R = (abcdefghi)$	$= R$
c	*d*	*e*	*f*	*g*	*h*	*i*	*a*	*b*	$R^2 = (acegibdfh)$	$= G$
d	*e*	*f*	*g*	*h*	*i*	*a*	*b*	*c*	$R^3 = (adg)(beh)(cfi)$	$= Y$
e	*f*	*g*	*h*	*i*	*a*	*b*	*c*	*d*	$R^4 = (aeidhcgbf)$	$= B$
f	*g*	*h*	*i*	*a*	*b*	*c*	*d*	*e*	$R^5 = (afbgchdie)$	$= B^{-1}$
g	*h*	*i*	*a*	*b*	*c*	*d*	*e*	*f*	$R^6 = (agd)(bhe)(cif)$	$= Y^{-1}$
h	*i*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	$R^7 = (ahfdbigec)$	$= G^{-1}$
i	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	$R^8 = (aihgfedcb)$	$= R^{-1}$

$R, G, B.$	$(1\ 9\text{gon})^3$	27
$Y.$	3 3gons	9
		36

9B

										4 colours.
a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	$1 = 1$	$= 1$
b	*c*	*a*	*e*	*f*	*d*	*h*	*i*	*g*	$R = (abc)(def)(ghi)$	$= R$
c	*a*	*b*	*f*	*d*	*e*	*i*	*g*	*h*	$R^2 = (acb)(dfe)(gih)$	$= R^{-}$
d	*e*	*f*	*g*	*h*	*i*	*a*	*b*	*c*	$G = (adg)(beh)(cfi)$	$= G$
e	*f*	*d*	*h*	*i*	*g*	*b*	*c*	*a*	$RG = (aei)(bfg)(cdh)$	$= B$
f	*d*	*e*	*i*	*g*	*h*	*c*	*a*	*b*	$R^2G = (afh)(bdi)(ceg)$	$= Y$
g	*h*	*i*	*a*	*b*	*c*	*d*	*e*	*f*	$G^2 = (agd)(bhe)(cif)$	$= G^{-1}$
h	*i*	*g*	*b*	*c*	*a*	*e*	*f*	*d*	$RG^2 = (ahf)(bid)(cge)$	$= Y^{-1}$
i	*g*	*h*	*c*	*a*	*b*	*f*	*d*	*e*	$R^2G^2 = (aie)(bgf)(chd)$	$= B^{-1}$

$R, G, B, Y.$	$(3\ 3\text{gons})^4$	36
		36

10A

											5 colours.
a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	1	$=1$
b	*c*	*d*	*e*	*f*	*g*	*h*		*j*	*a*	$R = (abcdefghij)$	$=R$
c	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*a*	*b*	$R^2 = (acegi)(bdfhj)$	$=G$
d	*e*	*f*	*g*	*h*	*i*	*j*	*a*	*b*	*c*	$R^3 = (adgjcfibeh)$	$=Y$
e	*f*	*g*	*h*	*i*	*j*	*a*	*b*	*c*	*d*	$R^4 = (aeicg)(bfdjh)$	$=B$
f	*g*	*h*	*i*	*j*	*a*	*b*	*c*	*d*	*e*	$R^5 = (af)(bg)(ch)(di)(ej)$	$=O$
g	*h*	*i*	*j*	*a*	*b*	*c*	*d*	*e*	*f*	$R^6 = (agcie)(bhjdf)$	$=B^{-1}$
h	*i*	*j*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	$R^7 = (ahebifcjgd)$	$=Y^{-1}$
i	*j*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	$R^8 = (aigec)(bjhfd)$	$=G^{-1}$
j	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	$R^9 = (ajihgfedcb)$	$=R^{-1}$

$R, G, Y, B.$ **(1 10gon)^4 40**
$O.$ **5 digons 5**
45

10B

											7 colours.
a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	$1 = 1$	$=1$
b	*c*	*d*	*e*	*a*	*j*	*f*	*g*	*h*	*i*	$R = (abcde)(fjihg)$	$=R$
c	*d*	*e*	*a*	*b*	*i*	*j*	*f*	*g*	*h*	$R^2 = (acebd)(figjh)$	$=Y$
d	*e*	*a*	*b*	*c*	*h*	*i*	*j*	*f*	*g*	$R^3 = (adbec)(fhjgi)$	$=Y^{-1}$
e	*a*	*b*	*c*	*d*	*g*	*h*	*i*	*j*	*f*	$R^4 = (aedcb)(fghij)$	$=R^{-1}$
f	*g*	*h*	*i*	*j*	*a*	*b*	*c*	*d*	*e*	$G = (af)(bg)(ch)(di)(ej)$	$=G$
g	*h*	*i*	*j*	*f*	*e*	*a*	*b*	*c*	*d*	$RG = (ag)(bh)(ci)(dj)(ef)$	$=B$
h	*i*	*j*	*f*	*g*	*d*	*e*	*a*	*b*	*c*	$R^2G = (ah)(bi)(cj)(df)(eg)$	$=O$
i	*j*	*f*	*g*	*h*	*c*	*d*	*e*	*a*	*b*	$R^3G = (ai)(bj)(cf)(dg)(eh)$	$=V$
j	*f*	*g*	*h*	*i*	*b*	*c*	*d*	*e*	*a*	$R^4G = (aj)(bf)(cg)(dh)(ei)$	$=I$

$R, Y.$ **(2 5gons)^2 20**
$G, B, O, V, I.$ **(5 digons)^5 25**
45

11A

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*
b	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*a*
c	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*a*	*b*
d	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*a*	*b*	*c*
e	*f*	*g*	*h*	*i*	*j*	*k*	*a*	*b*	*c*	*d*
f	*g*	*h*	*i*	*j*	*k*	*a*	*b*	*c*	*d*	*e*
g	*h*	*i*	*j*	*k*	*a*	*b*	*c*	*d*	*e*	*f*
h	*i*	*j*	*k*	*a*	*b*	*c*	*d*	*e*	*f*	*g*
i	*j*	*k*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*
j	*k*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*
k	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*

5 colours.

$1 = 1 = 1$ *R*, *G*, *Y*, *B*, *O*.   (1 11gon)^{5} 55

$R = (abcdefghijk) = R$

$R^{2} = (acegikbdfhj) = G$

$R^{3} = (adgjbehkcfi) = Y$

$R^{4} = (aeibfjcgkdh) = B$

$R^{5} = (afkejdichbg) = O$

$R^{6} = (agbhcidjekf) = O^{-1}$

$R^{7} = (ahdkgcjfbie) = B^{-1}$

$R^{8} = (aifckhebjgd) = Y^{-1}$

$R^{9} = (ajhfdbkigec) = G^{-1}$

$R^{10} = (akjihgfedcb) = R^{-1}$

55

12A

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*
b	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*	*a*
c	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*	*a*	*b*
d	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*	*a*	*b*	*c*
e	*f*	*g*	*h*	*i*	*j*	*k*	*l*	*a*	*b*	*c*	*d*
f	*g*	*h*	*i*	*j*	*k*	*l*	*a*	*b*	*c*	*d*	*e*
g	*h*	*i*	*f*	*k*	*l*	*a*	*b*	*c*	*d*	*e*	*f*
h	*i*	*j*	*k*	*l*	*a*	*b*	*c*	*d*	*e*	*f*	*g*
i	*j*	*k*	*l*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*
j	*k*	*l*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*
k	*l*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*
l	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*

6 colours.

$1 = 1 = 1$

$R = (abcdefghijkl) = R$

$R^{2} = (acegik)(bdfhjl) = G$

$R^{3} = (adgj)(behk)(cfil) = Y$

$R^{4} = (aei)(bfj)(cgk)(dhl) = B$

$R^{5} = (afkdibglejch) = O$

$R^{6} = (ag)(bh)(ci)(dj)(ek)(fl) = V$

$R^{7} = (ahcjelgbidkf) = O^{-1}$

$R^{8} = (aie)(bjf)(ckg)(dlh) = B^{-1}$

$R^{9} = (ajgd)(bkhe)(clif) = Y^{-1}$

$R^{10} = (akigec)(bljhfd) = G^{-1}$

$R^{11} = (alkjihgfedcb) = R^{-1}$

R, *O*.	(1 12gon)^{2}	24
G.	2 6gons	12
Y.	3 4gons	12
B.	4 3gons	12
V.	6 digons	6
		66

12*B*1

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*
b	*c*	*d*	*e*	*f*	*a*	*h*	*i*	*j*	*k*	*l*	*g*
c	*d*	*e*	*f*	*a*	*b*	*i*	*j*	*k*	*l*	*g*	*h*
d	*e*	*f*	*a*	*b*	*c*	*j*	*k*	*l*	*g*	*h*	*i*
e	*f*	*a*	*b*	*c*	*d*	*k*	*l*	*g*	*h*	*i*	*j*
f	*a*	*b*	*c*	*d*	*e*	*l*	*g*	*h*	*i*	*j*	*k*
g	*h*	*i*	*j*	*k*	*l*	*a*	*b*	*c*	*d*	*e*	*f*
h	*i*	*j*	*k*	*l*	*g*	*b*	*c*	*d*	*e*	*f*	*a*
i	*j*	*k*	*l*	*g*	*h*	*c*	*d*	*e*	*f*	*a*	*b*
j	*k*	*l*	*g*	*h*	*i*	*d*	*e*	*f*	*a*	*b*	*c*
k	*l*	*g*	*h*	*i*	*j*	*e*	*f*	*a*	*b*	*c*	*d*
l	*g*	*h*	*i*	*j*	*k*	*f*	*a*	*b*	*c*	*d*	*e*

7 colours.

$1 = 1 = 1$

$R = (abcdef)(ghijkl) = R$

$R^2 = (ace)(bdf)(gik)(hjl) = Y$

$R^3 = (ad)(be)(cf)(gj)(hk)(il) = B$

$R^4 = (aec)(bfd)(gki)(hlj) = Y^{-1}$

$R^5 = (afedcba)(glkjih) = R^{-1}$

$G = (ag)(bh)(ci)(dj)(ek)(fl) = G$

$RG = (ahcjel)(bidkfg) = P$

$R^2G = (aiegck)(bjfhdl) = O$

$R^3G = (aj)(bk)(cl)(dg)(eh)(fi) = V$

$R^4G = (akcgei)(bdlhfj) = O^{-1}$

$R^5G = (alejch)(bgfkdi) = P^{-1}$

R, P, O.	(2 6gons)^3	36
Y.	4 3gons	12
B, G, V.	(6 digons)^3	18
		66

12*B*2

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*
b	*c*	*d*	*e*	*f*	*a*	*l*	*g*	*h*	*i*	*j*	*k*
c	*d*	*e*	*f*	*a*	*b*	*k*	*l*	*g*	*h*	*i*	*j*
d	*e*	*f*	*a*	*b*	*c*	*j*	*k*	*l*	*g*	*h*	*i*
e	*f*	*a*	*b*	*c*	*d*	*i*	*j*	*k*	*l*	*g*	*h*
f	*a*	*b*	*c*	*d*	*e*	*h*	*i*	*j*	*k*	*l*	*g*
g	*h*	*i*	*j*	*k*	*l*	*a*	*b*	*c*	*d*	*e*	*f*
h	*i*	*j*	*k*	*l*	*g*	*f*	*a*	*b*	*c*	*d*	*e*
i	*j*	*k*	*l*	*g*	*h*	*e*	*f*	*a*	*b*	*c*	*d*
j	*k*	*l*	*g*	*h*	*i*	*d*	*e*	*f*	*a*	*b*	*c*
k	*l*	*g*	*h*	*i*	*j*	*c*	*d*	*e*	*f*	*a*	*b*
l	*g*	*h*	*i*	*j*	*k*	*b*	*c*	*d*	*e*	*f*	*a*

9 colours.

$1 = 1 = 1$

$R = (abcdef)(glkjih) = R$

$R^2 = (ace)(bdf)(gki)(hlj) = Y$

$R^3 = (ad)(be)(cf)(gj)(hk)(il) = B$

$R^4 = (aec)(bfd)(gik)(hjl) = Y^{-1}$

$R^5 = (afedcb)(ghijkl) = R^{-1}$

$G = (ag)(bh)(ci)(dj)(ek)(fl) = G$

$RG = (ah)(bi)(cj)(dk)(el)(fg) = P$

$R^2G = (ai)(bj)(ck)(dl)(eg)(fh) = O$

$R^3G = (aj)(bk)(cl)(dg)(eh)(fi) = V$

$R^4G = (ak)(bl)(cg)(dh)(ei)(fj) = I$

$R^5G = (al)(bg)(ch)(di)(ej)(fk) = S$

R.	2 6gons	12
Y.	4 3gons	12
B, G, P, O, V, I, S.	(6 digons)^7	42
		66

12*B*3

a	b	c	d	e	f	g	h	i	j	k	l
b	c	d	e	f	a	l	g	h	i	j	k
c	d	e	f	a	b	k	l	g	h	i	j
d	e	f	a	b	c	j	k	l	g	h	i
e	f	a	b	c	d	i	j	k	l	g	h
f	a	b	c	d	e	h	i	j	k	l	g
g	h	i	j	k	l	d	e	f	a	b	c
h	i	j	k	l	g	c	d	e	f	a	b
i	j	k	l	g	h	b	c	d	e	f	a
j	k	l	g	h	i	a	b	c	d	e	f
k	l	g	h	i	j	f	a	b	c	d	e
l	g	h	i	j	k	e	f	a	b	c	d

6 colours.

$1 = 1 = 1$

$R = (abcdef)(glkjih) = R$

$R^2 = (ace)(bdf)(gki)(hlj) = B$

$R^3 = (ad)(be)(cf)(gj)(hk)(il) = Y$

$R^4 = (aec)(bfd)(gik)(hjl) = B^{-1}$

$R^5 = (afedcb)(ghijkl) = R^{-1}$

$G = (agdj)(bhek)(cifl) = G$

$RG = (ahdk)(biel)(cjfg) = O$

$R^2G = (aidl)(bjeg)(ckfh) = P$

$R^3G = (ajdg)(bkeh)(clfi) = G^{-1}$

$R^4G = (akdh)(blei)(cgfj) = O^{-1}$

$R^5G = (aldi)(bgej)(chfk) = P^{-1}$

R.	**2 6gons**	**12**
G, O, P.	**$(3\ 4\text{gons})^3$**	**36**
B.	**4 3gons**	**12**
Y.	**6 digons**	**6**
		66

12*B*4

a	b	c	d	e	f	g	h	i	j	k	l
b	c	a	e	f	d	h	i	g	k	l	j
c	a	b	f	d	e	i	g	h	l	j	k
d	l	g	a	i	j	c	k	e	f	h	b
e	j	h	b	g	k	a	l	f	d	i	c
f	k	i	c	h	l	b	j	d	e	g	a
g	d	l	j	a	i	e	c	k	b	f	h
h	e	j	k	b	g	f	a	l	c	d	i
i	f	k	l	c	h	d	b	j	a	e	g
j	h	e	g	k	b	l	f	a	i	c	d
k	i	f	h	l	c	j	d	b	g	a	e
l	g	d	i	j	a	k	e	c	h	b	f

7 colours.

$1 = 1 = 1$

$R = (abc)(def)(ghi)(jkl) = R$

$R^2 = (acb)(dfe)(gih)(jlk) = R^{-1}$

$RGR^2 = (ad)(bl)(cg)(ei)(fj)(hk) = Y$

$RG = (aeg)(bjd)(chl)(fki) = B$

$RGR = (afl)(bkg)(cid)(ehj) = O$

$GR^2 = (age)(bdj)(clh)(fik) = B^{-1}$

$G = (ah)(be)(cj)(dk)(fg)(il) = G$

$GR = (aij)(bfh)(cke)(dlg) = P$

$R^2G = (aji)(bhf)(cek)(dgl) = P^{-1}$

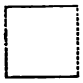
$R^2GR = (ak)(bi)(cf)(dh)(el)(dj) = V$

$R^2GR^2 = (alf)(bgk)(cdi)(ejh) = O^{-1}$

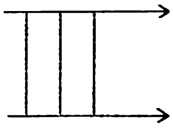
R, B, P, O.	**$(4\ 3\text{gons})^4$**	**48**
Y, G, V.	**$(6\ \text{digons})^3$**	**18**
		66

Extracting from these colourgroups the twocolourbonds contained in them respectively, we have the twocolourbonds shown in the annexed series of figures. I have in each case given the number $4B$, $6A$, etc., of the colourgroup in which the bond is contained, and which colourgroup is given conversely by the twocolourbond. The several points may have letters a, b, c, d, etc., attached to them at pleasure, but as the particular letters are quite immaterial, it seemed to me better to give the several figures without any letters.

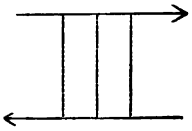
$4B$



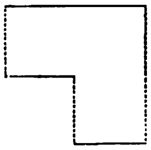
$6A$



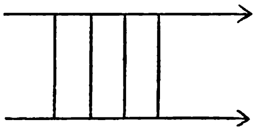
$6B$



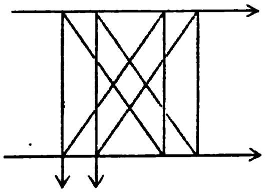
$6B$



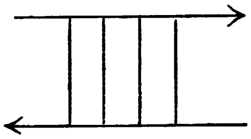
$8B1$



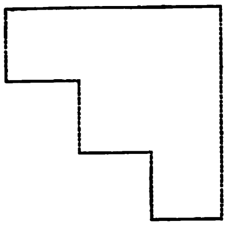
$8B1$



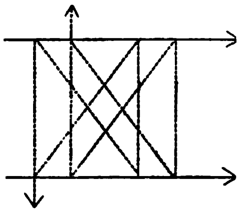
$8B2$



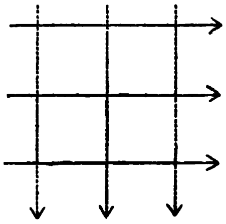
$8B2$



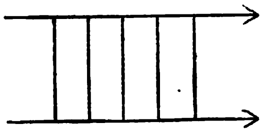
$8B3$



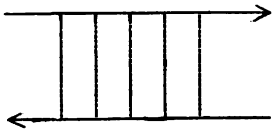
$9B$



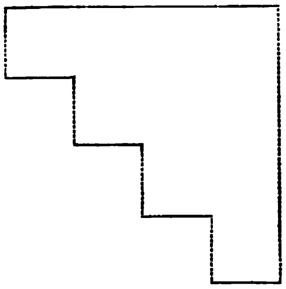
$10A$



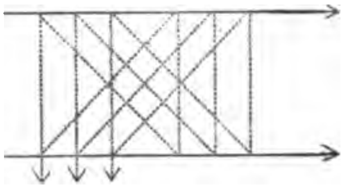
$10B$



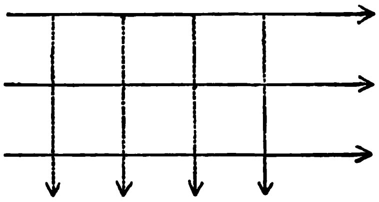
$10B$



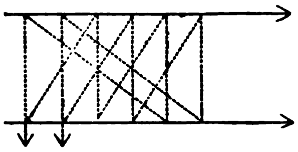
12*A*



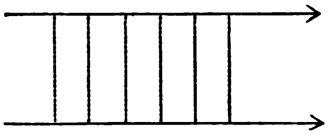
12*A*



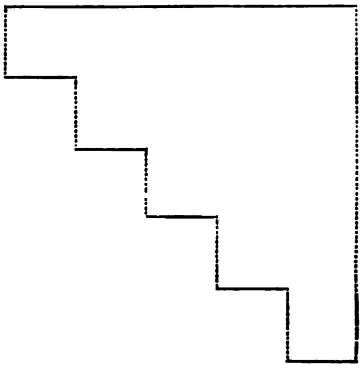
12*B*1



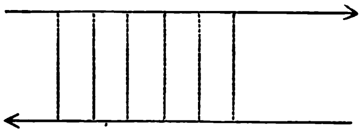
12*B*1



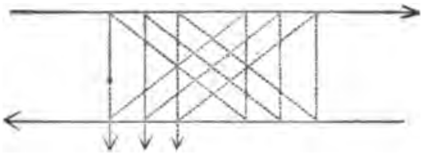
12*B*2



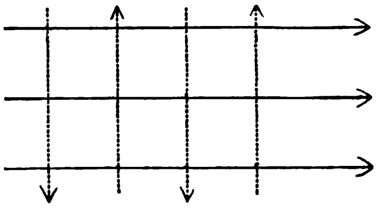
12*B*2



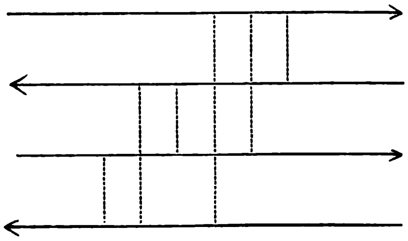
12*B*3



12*B*3

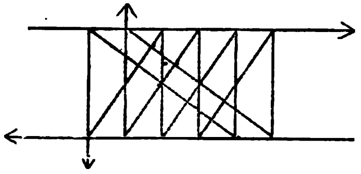


12*B*4

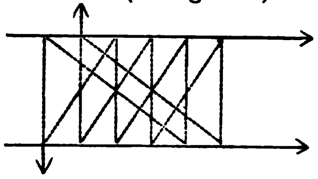


In any one of the foregoing forms of twocolourbond, each point is in its relations to the other points indistinguishable from each of the other points. This would seem to be a relation of symmetry equivalent to the before-mentioned condition that each route is of independent effect; and it would moreover seem as if the relation of symmetry were satisfied for each of the following forms:

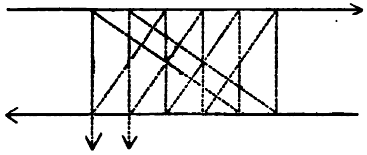
12 (wrong form).



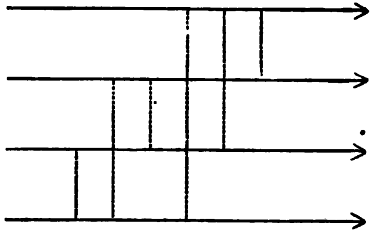
12 (wrong form).



12 (wrong form).



12 (wrong form).



Each of these is, however, a wrong form, not satisfying the condition that each route is of independent effect. As to this, observe that when the condition is satisfied, there are in all ($s =$) 12 non-equivalent routes, and there is thus a completely determinate square. When the condition is not satisfied, there are more than this number of non-equivalent routes, and there may very well be s routes giving rise to a latin square, viz. a square each line of which, and also each column of which, contains all the letters, and which thus seems at first sight to represent a substitution-group; but the substitutions by which each line of the square is derived from itself and the other lines of the square are not the same as those by which each line is derived from the top line, and thus the square does not represent a group. Thus in one of the above wrong forms, starting from the routes $R = (abcdef)(glkjih)$ and $G = (agciek)(bhdjfl)$, we have

12 (wrong form).

a	b	c	d	e	f	g	h	i	j	k	l	$1 \quad = 1$
b	c	d	e	f	a	l	g	h	i	j	k	$R \quad = (abcdef)(glkjih)$
c	d	e	f	a	b	k	l	g	h	i	j	$R^2 \quad = (ace)(bdf)(gki)(hlj)$
d	e	f	a	b	c	j	k	l	g	h	i	$R^3 \quad = (ad)(be)(cf)(gj)(hk)(il)$
e	f	a	b	c	d	i	j	k	l	g	h	$R^4 \quad = (aec)(bfd)(gik)(hjl)$
f	a	b	c	d	e	h	i	j	k	l	g	$R^5 \quad = (afedcb)(ghijkl)$
g	h	i	j	k	l	c	d	e	f	a	b	$G \quad = (agciek)(bhdjfl)$
h	i	j	k	l	g	b	c	d	e	f	a	$RG = (ahcjel)(bidkfg)$
i	j	k	l	g	h	a	b	c	d	e	f	$R^2G = (aickeg)(bjdlfh)$
j	k	l	g	h	i	f	a	b	c	d	e	$R^3G = (ajcleh)(bkdgfi)$
k	l	g	h	i	j	e	f	a	b	c	d	$R^4G = (akcgei)(bldhfj)$
l	g	h	i	j	k	d	e	f	a	b	c	$R^5G = (alchej)(bgdifk)$

	G	R^2	G
a	g	k	a
b	h	l	b
c	i	g	c
d	j	h	d
e	k	i	e
f	l	j	f
g	c	e	k
h	d	f	l
i	e	a	g
j	f	b	h
k	a	c	i
l	b	d	j

which is not a group; there is no substitution $G^{-1} = (akeicg)(blfjdh)$. And we see that in fact each route is not of independent effect; the route GR^2G leads as shown from the primitive arrangement *abcdefghijkl* to *abcdefklghij*, viz: it is a circuit from each of the points a, b, c, d, e, f, but not from any one of the remaining points g, h, i, j, k, l.

Vortex Motion in certain Triangles.

By A. E. H. Love, B.A., *Fellow of St. John's College, Cambridge.*

1. In the Proceedings of the London Mathematical Society, Vol. XII, Routh has explained a method whereby the motion of a fluid in two dimensions due to sources or vortices situated within a given region can be inferred from the motion due to sources or vortices of equal strength placed at the corresponding points of another region, whenever, by means of a transformation by conjugate functions, we can obtain a correlation* ("conforme Abbildung") of the first region with the second in such a way that elementary portions of the two regions are similar. The conditions for such correlation may be expressed as follows: Let Z be a function of a complex variable z, and suppose Z and z represented in the usual manner by points on two planes, then the boundary of any region in the z plane will be transformed into a certain curve in the Z plane. The function Z is to be chosen to be finite, continuous, and one-valued within the region in the z plane, and points on the boundary in the z plane are to correspond to points on the boundary in the Z plane in such a way that *one* point on either boundary corresponds to *one* point on the other.

If Φ and Ψ are the velocity- and stream-functions of any fluid motion in the Z plane, then $\Phi + \iota\Psi$ is a function of Z, and Routh shows that the velocity- and stream-functions in the corresponding motion within the correlated region of the Z plane can be inferred by simply substituting for Z in $\Phi + \iota\Psi$ its value expressed as a function of z.

In Art. 21 of Riemann's "Inaugural Dissertation" (Ges. Werke, p. 40) it is stated that the problem to find a correlation of two simply-connected plane regions has always one and only one solution,* and thus in particular it is always theoretically possible to correlate any given simply-connected plane region with that part of a plane which lies on one side of the axis of real quantities. In a memoir in Borchardt's Journal (Bd. 70), Schwarz has discussed the solution of the problem when the region to be correlated with the half plane is bounded by a polygon, and he has shown that when the polygon is a triangle in the z plane, the function Z is in general a transcendent having an infinite number of values at every point of the z plane, of which the value along any given branch of the function is, however, singly determinate (eindeutig) within the triangle. But there are certain exceptional cases in which Z is an algebraic function of an elliptic function $\wp z$ of z, $\wp z$ being one-valued at every point of the z plane. These exceptional cases are those of a triangle whose angles are

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* For the theory of the correlation of two plane regions the reader may consult Darboux, "Théorie Générale des Surfaces," ch. IV, or Holzmüller, "Einführung in die Theorie der isogonalen Verwandtschaften und der conformen Abbildungen."

$$\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right), \left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}\right), \left(\frac{2\pi}{3}, \frac{\pi}{6}, \frac{\pi}{6}\right), \text{ or } \left(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}\right).$$

The integration of the differential equation on which the correlation depends had been considered by Briot and Bouquet in their "Théorie des Fonctions doublement périodiques."†

In what follows I propose to work out in detail these cases, and to apply the solution to the problem of the motion of a single vortex within the triangle. The solution for any number of vortices can be found by summation, while that for sources is obtained by interchanging Φ and Ψ.

2. Schwarz shows that if a, b, c are points on the real axis in the Z plane, then a triangle whose angles are $\alpha\pi$, $\beta\pi$, $\gamma\pi$ in the z plane can be correlated with the half plane above the real axis in the Z plane by means of the equation

$$\frac{d}{dZ}\left(\log \frac{dz}{dZ}\right) = \frac{\alpha - 1}{Z - a} + \frac{\beta - 1}{Z - b} + \frac{\gamma - 1}{Z - c},$$

and the corners of the triangle will correspond to the points $Z = a, b, c$.

This appears because (1) the function z of Z is plainly in any correlation of the form $C_1 z + C_2$, where C_1, C_2 are arbitrary constants, and z is one determination of the function; (2) the transformation

$$\frac{d}{dZ}\left(\log \frac{dz}{dZ}\right) = \frac{\alpha - 1}{Z - a}$$

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*All solutions are counted as one solution in which *finite* triangles remain similar.

†See the same author's "Théorie des Fonctions Elliptiques," pp. 388 seq.

effects a correlation of the half plane Z (imaginary part positive) with the space bounded by two lines in the z plane inclined to each other at angle $\alpha\pi$, and meeting in the point corresponding to $Z = a$.

It follows that the desired correlation is effected by means of the equation

$$C_1 z + C_2 = \int^z (Z-a)^{\alpha-1}(Z-b)^{\beta-1}(Z-c)^{\gamma-1} dZ, \tag{A}$$

in which the points a, b, c are taken to lie in a definite order on the real axis, viz. if A, B, C be the order of the corners of the triangle in the z plane at which the angles are $\alpha\pi$, $\beta\pi$, $\gamma\pi$ when its contour is described in the positive sense, *i. e.* with its area on the left, then a, b, c is to be the order of the points in the Z plane which correspond to the corners, the real axis being described in the positive sense.

CASE I.—*The equilateral triangle.*

3. In this case the equation (A) becomes

$$C_1 z + C_2 = \int^z \frac{dZ}{(Z-a)^{\frac{2}{3}}(Z-b)^{\frac{2}{3}}(Z-c)^{\frac{2}{3}}}. \tag{1}$$

We may take $a = 0$, $b = 1$, $c = -1$; then, supposing $Z = -\frac{1}{Z'}$, we have

$$C_1 z + C_2 = \int^{z'} \frac{dZ'}{(Z'^3 - 1)^{\frac{2}{3}}}.$$

Write

$$1 - Z'^3 = x^3 = 1 - \frac{1}{Z^2}, \tag{2}$$

thus

$$C_1 z + C_2 = -\frac{3}{2}\int^x \frac{dx}{\sqrt{1-x^3}},$$

or

$$-\frac{1}{3}(C_1 z + C_2) = \int^x \frac{dx}{\sqrt{4(x^3-1)}}.$$

Hence, adjusting constants, we may write

$$x = \wp z, \tag{3}$$

where $\wp z$ is defined by the equation

$$\left.\begin{aligned} \wp'^2 z &= 4(\wp^3 z - 1) = 4(\wp z - e_1)(\wp z - e_2)(\wp z - e_3), \\ e_1 &= -\frac{1}{2} + \iota\frac{\sqrt{3}}{2},\ e_2 = 1,\ e_3 = -\frac{1}{2} - \iota\frac{\sqrt{3}}{2}. \end{aligned}\right\} \tag{4}$$

The correlation is effected by means of the equation

$$1-\frac{1}{Z^3}=\wp^3 z,$$

giving $$Z^3=\frac{1}{1-\wp^3 z}=\frac{-1}{(\wp z-e_1)(\wp z-e_2)(\wp z-e_3)},$$

thus $$-\frac{\iota Z}{2}=\frac{1}{\wp' z}. \tag{5}$$

4. The elliptic functions given by (4) have been discussed by Greenhill.* We have to notice that the discriminant is negative and equal to -432. The real half-period ω_2 is given by the equation

$$\omega_2=\int_1^\infty \frac{dx}{\sqrt{4(x^3-1)}}=\frac{1}{6}\,\frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2}{3}\right)},$$

the corresponding imaginary half-period $\omega_2'=\iota\omega_2\sqrt{3}$. The periods ω_1, ω_3 corresponding to e_1, e_3 are $\omega_1=\frac{1}{2}(\omega_2-\omega_2')$, $\omega_3=\frac{1}{2}(\omega_2+\omega_2')$.

We can also express in a simple form all the roots of the function $\wp z$. We know that for pure imaginary values of z, $\wp z$ passes through all real values between $e_2\,(=1)$ and $-\infty$, and takes the same value at points whose z differ by multiples of $2\omega_2'$. Hence there is a root for some value of z a pure imaginary between 0 and $2\omega_2'$. Now we may show that if $\wp z=0$, then $\wp(3z)=\infty$, this follows at once from the formula

$$\wp 3z-\wp z=-\frac{\wp'^2 z\,(2\wp^6 z-40\wp^3 z-16)}{3\,(\wp^4 z-4\wp z)}$$

by putting $\wp z=0$. Thus it appears that $\frac{2}{3}\omega_2'$ is a root of $\wp z$ which is pure imaginary and lies between 0 and $2\omega_2'$, the other root $\frac{4}{3}\omega_2'$ may be regarded as $-\frac{2}{3}\omega_2'+2\omega_2'$, so that it differs from the former by a period and a change of sign.

5. Returning to equation (5), we see that $\wp^3 z=0$ gives the values of z which correspond to $Z=\pm 1$, and $\wp z=\infty$ corresponds to $Z=0$. We may choose

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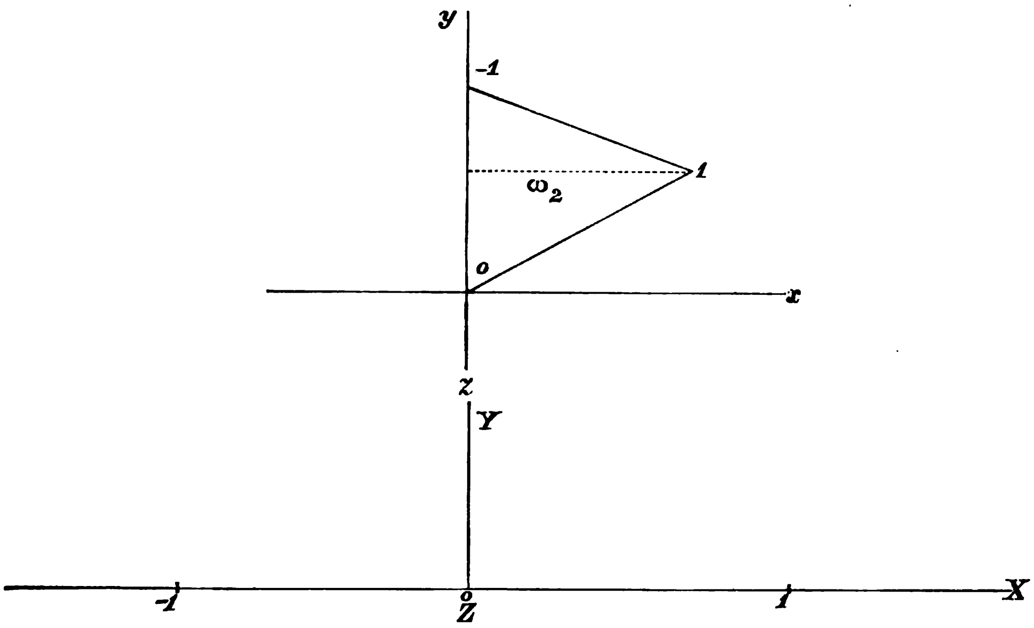
[*] "The Trajectory for the Cubic Law of Resistance," Proc. Royal Artillery Institution, 1886.

the pole z for $Z=0$ to be $z=0$, and the root z for $Z=-1$ to be $z=\frac{2}{3}\omega_2'$; then since $-\frac{2}{3}\omega_2'=\left(\omega_2+\frac{1}{3}\omega_2'\right)-2\omega_3$, we have $\wp\left(\omega_2+\frac{1}{3}\omega_2'\right)=0$, and $z=\omega_2+\frac{1}{3}\omega_2'$ may be taken to correspond to $Z=1$.

The function $Z=2\iota/\wp' z$ is finite, continuous, and one-valued within the equilateral triangle in the z plane whose corners have coordinates

$$(0,0),\ \left(0,\frac{2}{\sqrt{3}}\omega_2\right),\ \left(\omega_2,\frac{1}{\sqrt{3}}\omega_2\right),$$

and the points on the real axis in the Z plane correspond to points on the sides of the triangle, the points $Z=0,\ 1,\ -1$ being those which correspond to the corners. The height of the triangle is the real half-period ω_2.



The figure represents the boundaries of the two correlated regions, the points $0,\ 1,\ -1$ correspond to the angular points of the triangle and these are marked with the same numbers in both figures.

6. Now let z_0 be the point at which is a vortex of strength m, and let Z_0 be the corresponding point of the Z plane, the imaginary part of Z_0 being supposed

positive. Then if Z'_0, z'_0 be the conjugate imaginaries of Z_0, z_0, the fluid motion is given by

$$\Psi + \iota\Phi = \frac{m}{2\pi}\log\frac{Z-Z_0}{Z-Z'_0} = \frac{m}{2\pi}\log\frac{\wp' z - \wp' z_0}{\wp' z - \wp' z'_0} \tag{6}$$

The position of z_0 is given by $\wp^3 z_0 = 1 - \frac{1}{Z_0^2}$, and if z_0 be one of the roots of this equation, there will be other roots which do not differ from z_0 by multiples of the periods. There are in fact two such roots z_1, z_2 for which $\wp z_1 = e_1 \wp z_0$, $\wp z_2 = e_3 \wp z_0$, and then $z_1 = e_1 z_0$ and $z_2 = e_3 z_0$.

This appears from the equation

$$\left(\frac{d\,(e_1 \wp z)}{d\,(e_1 z)}\right)^2 = 4\,(\wp^3 z - 1) = 4\,\{(e_1 \wp z)^3 - 1\}$$

e_1 being a cube root of unity, from which we deduce

so
$$\left.\begin{aligned} \wp e_1 z &= e_1 \wp z \\ \wp e_3 z &= e_3 \wp z \end{aligned}\right\} \tag{7}$$[*]

The roots z_0, z_1, z_2 of the equation $\wp^3 z = 1 - \frac{1}{Z_0^2}$ satisfy the relation $z_0 + z_1 + z_2 = 0$. If z'_0 be the conjugate imaginary to z_0, then the conjugate imaginaries z'_1, z'_2 to z_1, z_2 are $e_3 z'_0$, $e_1 z'_0$.

Now consider the function

$$\phi_2(z) = \begin{vmatrix} 1 & \wp z & \wp' z \\ 1 & \wp z_1 & \wp' z_1 \\ 1 & \wp z_2 & \wp' z_2 \end{vmatrix} = (\wp z_2 - \wp z_1)(\wp' z - \wp' z_0), \tag{8}$$

where $\wp' z_1 = \wp' z_2 = \wp' z_0$.

We know that

$$\phi_2(z) = 2\,\frac{\sigma(z_1 - z)\,\sigma(z_2 - z)\,\sigma(z + z_1 + z_2)\,\sigma(z_2 - z_1)}{\sigma^3 z\,\sigma^3 z_1\,\sigma^3 z_2}\dagger\,. \tag{9}$$

Hence, remembering that σz is an odd function, and that

$$\wp z_2 - \wp z_1 = -\frac{\sigma(z_2 + z_1)\,\sigma(z_2 - z_1)}{\sigma^2 z_2\,\sigma^2 z_1}, \tag{10}$$

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[*] Greenhill, loc. cit. Art. 6.

† Halphen, "Traité des Fonctions Elliptiques," p. 219.

we obtain

$$\wp' z - \wp' z_0 = 2\,\frac{\sigma(z-z_0)\,\sigma(z-z_1)\,\sigma(z-z_2)}{\sigma^3 z\,\sigma z_0\,\sigma z_1\,\sigma z_2}\,, \tag{11}$$

so that the fluid motion is given by

$$\Psi + \iota\Phi = \frac{m}{2\pi}\log\frac{\sigma(z-z_0)\,\sigma(z-z_1)\,\sigma(z-z_2)}{\sigma(z-z_0')\,\sigma(z-z_1')\,\sigma(z-z_2')}\,. \tag{12}$$

7. Greenhill* has considered this case of fluid motion proceeding by the method of images. He observes that the images are

$$e_1 z_0,\ e_3 z_0,\ z_0',\ e_1 z_0',\ e_3 z_0'$$

and the similar sets grouped round centres of hexagons whose coordinates are $2m\omega_2$, $2m\omega_2\sqrt{3}$; $(2m+1)\,\omega_2$, $(2m+1)\,\omega_2\sqrt{3}$. As the vectors of all these are included in the formula $2m\omega_1 + 2m'\omega_3$, the solution found by summation will be that in (12), since

$$\sigma z = z\Pi_w'\left(1-\frac{z}{w}\right)e^{\frac{z}{w}+\frac{1}{2}\frac{z^2}{w^2}},\dagger$$

where w has all the values taken by the quantity $2m\omega_1 + 2m'\omega_3$, when m, m' are integers varying from 0 to ± ∞, except the value 0. In Greenhill's solution, which differs from (12) in form, the σ functions are formed with ω_2 and ω_2' as fundamental half-periods, and, since $\omega_2 + \omega_2'$ is a period, the σ_2 so formed is included in the σ formed with ω_1, ω_3 as fundamental half-periods.

CASE II.—*The right-angled triangle containing an angle of* 60°.

8. In this case equation (A) becomes

$$C_1 z + C_2 = \int^z \frac{dZ}{(Z-a)^{\frac{1}{2}}(Z-b)^{\frac{2}{3}}(Z-c)^{\frac{5}{6}}}\,. \tag{13}$$

Take $a=0$, $b=1$, $c=-1$, and suppose $z=\dfrac{1}{Z'}$, then

$$C_1 z + C_2 = -\int^{Z'} \frac{dZ'}{(1-Z')^{\frac{2}{3}}(1+Z')^{\frac{5}{6}}}\,.$$

Write

$$\frac{Z'-1}{Z'+1} = x^3 = \frac{1-Z}{1+Z}, \tag{14}$$

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[*] "Applications of Weierstrass's Elliptic Functions," Proc. Lond. Math. Soc., Vol. XVIII, p. 377.

[†] Schwarz, "Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen," p. 5.

then

$$C_1 z + C_2 = -\frac{3}{2}\int^x \sqrt{\frac{2}{1-x^3}}\,dx$$

$$= 3\sqrt{2}\,\iota \int^x \frac{dx}{4(x^3-1)}. \tag{15}$$

Hence, adjusting constants,

$$x = \wp z, \tag{16}$$

where the invariants and periods of the elliptic functions are the same as in Case I.

The correlation is effected by means of the equation

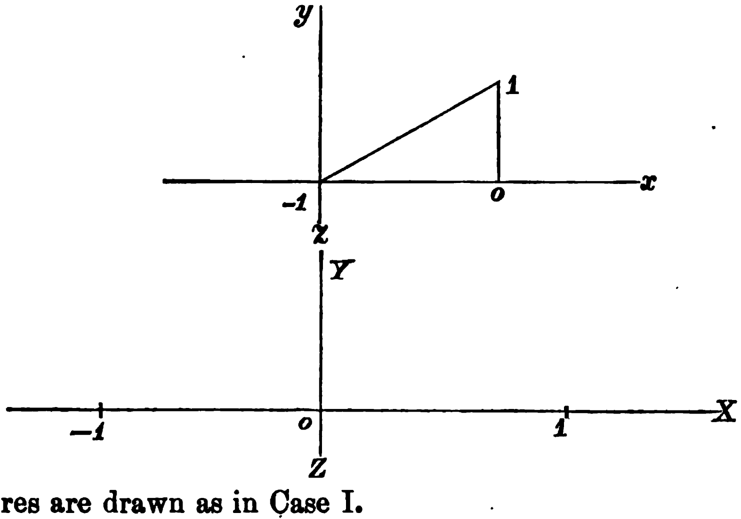
$$\frac{1-Z}{1+Z} = \wp^3 z,$$

or

$$Z = \frac{1-\wp^3 z}{1+\wp^3 z}. \tag{17}$$

The point $Z=0$ corresponds to $\wp^3 z = 1$ or $\wp' z = 0$, giving $z = \omega_1, \omega_2$, or ω_3, we may choose $z = \omega_2$. The point $Z = 1$ corresponds to $\wp^3 z = 0$, we may choose $z = \omega_2 + \frac{1}{3}\omega_2'$. The point $Z = -1$ corresponds to $\wp^3 z = \infty$, and we may choose $z = 0$.

The function $Z = (1-\wp^3 z)/(1+\wp^3 z)$ is finite, continuous, and one-valued within the right-angled triangle in the z plane whose corners have coordinates $(0, \omega_2)$, $\left(\omega_2, \frac{1}{\sqrt{3}}\omega_2\right)$, $(0, 0)$, and the points on the real axis in the Z plane correspond to the points on the sides of the triangle, the points $Z = 0, 1, -1$ being those which correspond to the corners. The side opposite to the angle of 60^c is of length equal to the real half-period ω_2.



The figures are drawn as in Case I.

9. Suppose there is a vortex of strength m at z_0, and let Z_0 be the corresponding point in the Z plane, and z_0', Z_0' the conjugate imaginaries to z_0, Z_0, then the fluid motion is given by

$$\Psi + \iota\Phi = \frac{m}{2\pi}\log\frac{Z-Z_0}{Z-Z_0'} = \frac{m}{2\pi}\log\frac{\wp^3 z - \wp^3 z_0}{\wp^3 z - \wp^3 z_0'}\,. \tag{18}$$

The roots of $\wp^3 z = \wp^3 z_0$ are z_0, $e_1 z_0$, $e_2 z_0$, say z_0, z_1, z_2, and those of $\wp^3 z = \wp^3 z_0'$ are the conjugate imaginaries, say z_0', z_1', z_2', resolving the numerator and denominator of (18) into factors and remembering the fundamental equation (10), we find

$$\Psi + \iota\Phi = \frac{m}{2\pi}\log\frac{\sigma(z-z_0)\,\sigma(z-z_1)\,\sigma(z-z_2)\,\sigma(z+z_0)\,\sigma(z+z_1)\,\sigma(z+z_2)}{\sigma(z-z_0')\,\sigma(z-z_1')\,\sigma(z-z_2')\,\sigma(z+z_0')\,\sigma(z+z_1')\,\sigma(z+z_2')}\,. \tag{19}$$

CASE III.—*The isosceles triangle containing an angle of* 120°.

10. In this case the equation (A) becomes

$$C_1 z + C_2 = \int^z \frac{dZ}{(Z-a)^{\frac{1}{3}}(Z-b)^{\frac{5}{6}}(Z-c)^{\frac{5}{6}}}\,. \tag{20}$$

Take $a=0$, $b=1$, $c=-1$, and suppose $z=\dfrac{1}{Z'}$, then

$$C_1 z + C_2 = \int^{z'} - \frac{dZ'}{(1-Z'^2)^{\frac{5}{6}}}\,.$$

Write

$$x^3 = \frac{1}{1-Z'^2} = \frac{Z^2}{Z^2-1}, \tag{21}$$

then

$$C_1 z + C_2 = -3\int^x \frac{dx}{\sqrt{4(x^3-1)}}\,. \tag{22}$$

Hence, adjusting constants,

$$x = \wp z, \tag{23}$$

where the invariants and periods of the elliptic functions are the same as in Case I.

The correlation is effected by means of the equation

$$Z^2 = \frac{\wp^3 z}{\wp^3 z - 1},$$

or

$$Z = 2\,\frac{\wp^{\frac{3}{2}} z}{\wp' z}\,. \tag{24}$$

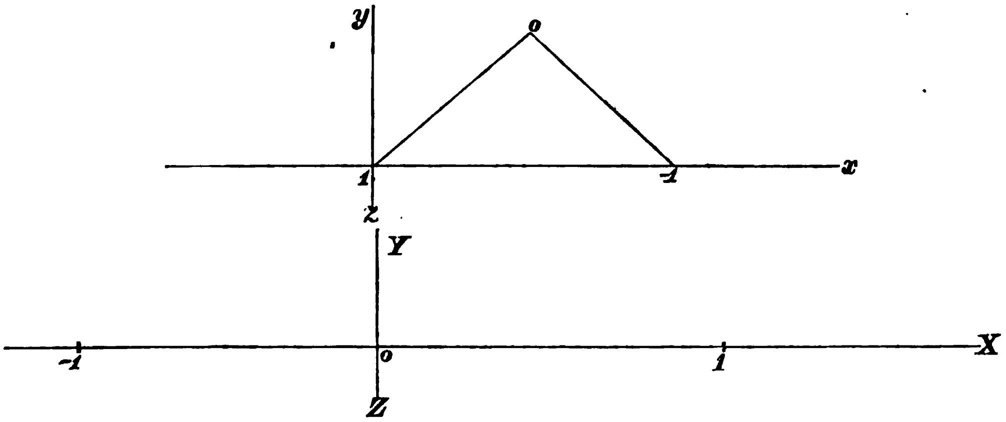
The point $Z=0$ corresponds to $\wp z = 0$ and we may choose $z = \omega_2 + \frac{1}{3}\omega_2'$. The points $Z = \pm 1$ correspond to $\wp z = \infty$, and we may choose $z=0$ for $Z=1$

and $z = 2\omega_2$ for $Z = -1$. Then since the function $\wp z$ does not vanish anywhere within the triangle whose corners are $z = \omega_2 + \frac{1}{3}\omega_2'$, 0, $2\omega_2$, it follows that the function $\wp^{\frac{1}{2}}z$ has no branch-point within the triangle, and is consequently finite, continuous, and one-valued within the triangle when its value has been chosen for one point.

Thus, choosing one branch of the function $\wp^{\frac{1}{2}}z$, we may say that the function $Z = 2\wp^{\frac{1}{2}}z/\wp'z$ is finite, continuous, and one-valued within the isosceles triangle in the z plane whose corners have coordinates

$$\left(\omega_2,\ \frac{1}{\sqrt{3}}\omega_2\right),\ (0,\ 0),\ (2\omega_2,\ 0),$$

and the points on the real axis in the Z plane correspond to the points on the sides of the triangle, the points $Z = 0,\ 1,\ -1$ being those which correspond to the corners. The length of the base of the triangle is the real period $2\omega_2$.



The figures are drawn as in Case I.

11. Suppose there is a vortex of strength m at z_0, and let z_0' be the conjugate imaginary, Z_0, Z_0' the corresponding points in the Z plane. Then the fluid motion is given by

$$\Psi + \iota\Phi = \frac{m}{2\pi}\log\frac{Z - Z_0}{Z - Z_0'} = \frac{m}{2\pi}\log\left\{\left(\frac{\wp^{\frac{1}{2}}z}{\wp'z} - \frac{\wp^{\frac{1}{2}}z_0}{\wp'z_0}\right)\Big/\left(\frac{\wp^{\frac{1}{2}}z}{\wp'z} - \frac{\wp^{\frac{1}{2}}z_0'}{\wp'z_0'}\right)\right\} \quad (25)$$

The function $\frac{\wp^{\frac{1}{2}}z}{\wp'z} - \frac{\wp^{\frac{1}{2}}z_0}{\wp'z_0}$ is not a rational function of $\wp z$, $\wp'z$ and does not appear to be expressible as a product of σ functions.

CASE IV.—*The isosceles right-angled triangle.*

12. In this case the equation (A) becomes

$$C_1 z + C_2 = \int^z \frac{dZ}{(Z-a)^{\frac{1}{2}}(Z-b)^{\frac{3}{4}}(Z-c)^{\frac{3}{4}}}. \tag{26}$$

Take $a=0$, $b=1$, $c=-1$ and suppose $Z=\frac{1}{Z'}$, then

$$C_1 z + C_2 = -\int^{z'} \frac{dZ'}{(1-Z'^2)^{\frac{3}{4}}}.$$

Write

$$\frac{1}{1-Z'^2} = x^2 = \frac{Z^2}{Z^2-1}, \tag{27}$$

then

$$C_1 z + C_2 = -\int \frac{dx}{\sqrt{x(x^2-1)}}. \tag{28}$$

Hence, adjusting constants,

$$x = \wp z, \tag{29}$$

where $\wp z$ is defined by the equation

$$\left.\begin{aligned} \wp'^2 z &= 4(\wp z - e_1)(\wp z - e_2)(\wp z - e_3), \\ e_1 &= 1,\ e_2 = 0,\ e_3 = -1. \end{aligned}\right\} \tag{30}$$

The correlation is effected by means of the equation

$$\frac{Z^2}{Z^2-1} = \wp^2 z,$$

giving

$$Z = \frac{\wp z}{\sqrt{\wp^2 z - 1}} = \frac{\wp z - e_2}{\sqrt{(\wp z - e_1)(\wp z - e_3)}}. \tag{31}$$

15. In the elliptic functions given by (30) we have to notice that the discriminant is positive, and the invariant $\wp_3 = 0$. The real half-period ω is given by

$$\omega = \int_1^\infty \frac{dx}{\sqrt{4x(x^2-1)}} = \frac{1}{4}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)},$$

and the pure imaginary half-period $\omega' = \iota\omega$.

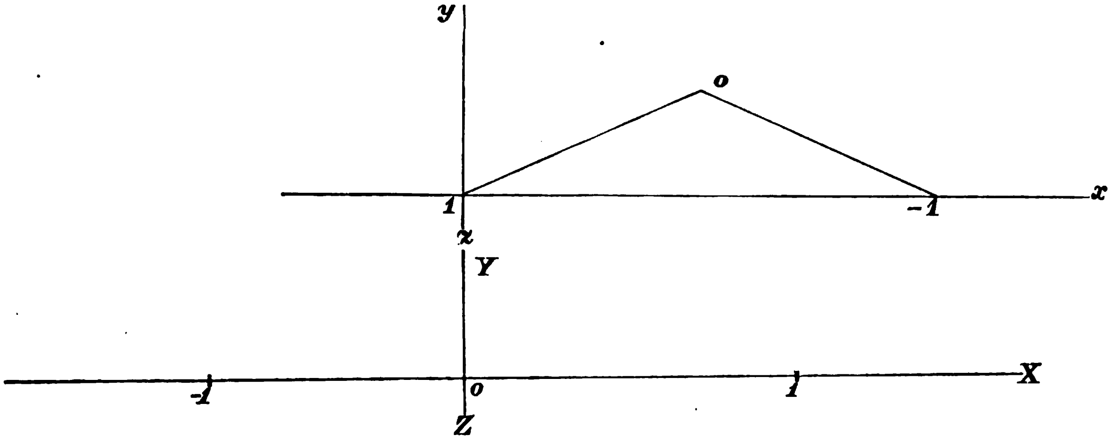
One root of $\wp z$ is easily found since $\wp(\omega+\omega') = e_2 = 0$.

Again $\sqrt{\wp z - e_1}$ and $\sqrt{\wp z - e_3}$ are singly determinate functions, viz. $\sqrt{\wp z - e_1} = \frac{\sigma_1 z}{\sigma z}$, and $\sqrt{\wp z - e_3} = \frac{\sigma_3 z}{\sigma z}$, and since $e_2 = 0$ we have in like manner $\wp z = \frac{\sigma_2^2 z}{\sigma^2 z}$. Hence

$$Z = \frac{\sigma_2^2 z}{\sigma_1 z \sigma_3 z}. \tag{32}$$

14. The point $Z = 0$ corresponds to $\wp z = 0$ and we may choose $z = \omega + \omega'$. The points $Z = \pm 1$ correspond to $\wp z = \infty$, and we may choose $z = 0$ for $Z = 1$, and $z = 2\omega$ for $Z = -1$.

The function $Z = \sigma_2^2 z / (\sigma_1 z \sigma_3 z)$ is finite, continuous, and one-valued within the isosceles right-angled triangle in the z plane whose corners have coordinates (ω, ω), $(0, 0)$, $(2\omega, 0)$, and points on the real axis in the Z plane correspond to points on the sides of the triangle, the points $Z = 0, 1, -1$ being those which correspond to the corners. The length of the hypothenuse is the real period 2ω.



The figures are drawn as in Case I.

15. Suppose there is a vortex of strength m at the point z_0, and let Z_0 be the corresponding point in the Z plane, Z_0' the image of Z_0 in the real axis and z_0' the corresponding point to Z_0', it will be necessary to prove that z_0, z_0' are conjugate imaginaries. We shall show that if z_0' be the conjugate imaginary to z_0, then Z_0' is conjugate to Z_0.

Since η* and ω are real, we see that the function $\sigma_1(\alpha + \iota\beta)$, being equal to $\frac{\sigma(\alpha + \iota\beta + \omega)}{\sigma\omega} e^{-\eta(\alpha + \iota\beta)}$, is the conjugate imaginary to $\sigma_1(\alpha - \iota\beta)$.

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* η is the constant $\frac{\sigma'\omega}{\sigma\omega}$ which $= \frac{1}{\omega} + \int_0^{\iota\omega} \left(\frac{1}{x^2} - \wp x \right) dx$. The constant η' is $\frac{\sigma'\omega'}{\sigma\omega'}$, and the relations $\eta\omega' - \eta'\omega = \frac{1}{2}\iota\pi$, $\omega' = \iota\omega$, $\eta' = -\iota\eta$ have place (Halphen, ch. V).

The function $\sigma_2(\alpha+\iota\beta)=\dfrac{\sigma\{\alpha+\omega+\iota(\beta+\omega)\}}{\sigma(\omega+\iota\omega)}\,e^{-\eta(\alpha+\iota\beta)+\iota\eta(\alpha+\iota\beta)}$, and the exponential factor is $e^{-\eta(\alpha+\beta)-\iota\eta(\beta-\alpha)}$. Also

$$\begin{aligned}\sigma_2(\alpha-\iota\beta)&=\frac{\sigma\{(\alpha+\omega-\iota(\beta+\omega)+2\omega')\}}{\sigma(\omega-\iota\omega+2\omega')}\,e^{-\eta(\alpha-\beta)+\iota\eta(\beta+\alpha)}\\&=\frac{\sigma\{(\alpha+\omega)-\iota(\beta+\omega)\}}{\sigma(\omega-\iota\omega)}\,e^{2\eta'(\alpha-\iota\beta)+\iota\eta(\beta-\alpha)}\\&=\frac{\sigma\{(\alpha+\omega)-\iota(\beta+\omega)\}}{\sigma(\omega-\iota\omega)}\,e^{-\eta(\alpha+\beta)+\iota\eta(\beta-\alpha)}.\end{aligned}$$

Thus $\sigma_2(\alpha-\iota\beta)$ and $\sigma_2(\alpha+\iota\beta)$ are conjugate imaginaries.

Again, the function $\sigma_3(\alpha+\iota\beta)=\dfrac{\sigma\{\alpha+\iota(\beta+\omega)\}}{\sigma(\iota\omega)}\,e^{\iota\eta(\alpha+\iota\beta)}$ and the exponential factor is $e^{-\eta(\beta-\iota\alpha)}$. Also

$$\begin{aligned}\sigma_3(\alpha-\iota\beta)&=\frac{\sigma\{\alpha-\iota(\beta+\omega)+2\omega'\}}{\sigma(-\iota\omega+2\omega')}\,e^{\eta(\beta+\iota\alpha)}\\&=\frac{\sigma\{\alpha-\iota(\beta+\omega)\}}{\sigma(-\iota\omega)}\,e^{2\eta'(\alpha-\iota\beta)+\eta(\beta+\iota\alpha)}\\&=\frac{\sigma\{\alpha-\iota(\beta+\omega)\}}{\sigma(-\iota\omega)}\,e^{-\eta(\beta+\iota\alpha)}.\end{aligned}$$

Thus $\sigma_3(\alpha+\iota\beta)$ and $\sigma_3(\alpha-\iota\beta)$ are conjugate imaginaries, and hence it follows that z_0, z_0' being conjugate imaginaries, Z_0, Z_0' are so, and inversely.

16. The fluid motion is given by

$$\Psi+\iota\Phi=\frac{m}{2\pi}\log\frac{Z-Z_0}{Z-Z_0'}=\frac{m}{2\pi}\log\left(\frac{\sigma_2^2z}{\sigma_1z\sigma_3z}-\frac{\sigma_2^2z_0}{\sigma_1z_0\sigma_3z_0}\right)\Big/\left(\frac{\sigma_2^2z}{\sigma_1z\sigma_3z}-\frac{\sigma_2^2z_0'}{\sigma_1z_0'\sigma_3z_0'}\right).\quad(33)$$

Remarking that $\qquad \wp z-1=\sigma_1^2z/\sigma^2z,$

and $\qquad \wp z+1=\sigma_3^2z/\sigma^2z,$

so that $\qquad \sigma_1^2z+\sigma_3^2z=2\sigma^2z\wp z=2\sigma^2z,$

equation (33) becomes

$$\Psi+\iota\Phi=\frac{m}{2\pi}\log\frac{(\sigma_1z\sigma_1z_0-\sigma_3z\sigma_3z_0)(\sigma_1z\sigma_3z_0-\sigma_3z\sigma_1z_0)}{(\sigma_1z\sigma_1z_0'-\sigma_3z\sigma_3z_0')(\sigma_1z\sigma_3z_0'-\sigma_3z\sigma_1z_0')}.\qquad(34)$$

The factor $\sigma_1z\sigma_3z_0-\sigma_1z_0\sigma_3z$ of the numerator may be put into factors. Using

Schwarz's formula [D][9], p. 81, and taking first $\mu = 1$, $\lambda = 3$, and then $\mu = 3$, $\lambda = 1$, we have

$$\sigma_1 z \sigma_3 z_0 = \sigma_3 \frac{z+z_0}{2} \sigma_1 \frac{z+z_0}{2} \sigma_3 \frac{z-z_0}{2} \sigma_1 \frac{z-z_0}{2}$$
$$- (e_1 - e_3) \sigma \frac{z+z_0}{2} \sigma_3 \frac{z+z_0}{2} \sigma \frac{z-z_0}{2} \sigma_3 \frac{z-z_0}{2},$$

$$\sigma_1 z_0 \sigma_3 z = \sigma_1 \frac{z+z_0}{2} \sigma_3 \frac{z+z_0}{2} \sigma_1 \frac{z-z_0}{2} \sigma_3 \frac{z-z_0}{2}$$
$$- (e_3 - e_1) \sigma \frac{z+z_0}{2} \sigma_3 \frac{z+z_0}{2} \sigma \frac{z-z_0}{2} \sigma_3 \frac{z-z_0}{2},$$

whence

$$\sigma_1 z \sigma_3 z_0 - \sigma_1 z_0 \sigma_3 z = 2(e_1 - e_3) \sigma \frac{z+z_0}{2} \sigma \frac{z-z_0}{2} \sigma_3 \frac{z+z_0}{2} \sigma_3 \frac{z-z_0}{2}.$$

So that the fluid motion is given by

$$\Psi + \iota\Phi = \frac{m}{2\pi} \log \frac{(\sigma_1 z \sigma_1 z_0 - \sigma_3 z \sigma_3 z_0) \sigma \frac{1}{2}(z+z_0) \sigma \frac{1}{2}(z-z_0) \sigma_2 \frac{1}{2}(z+z_0) \sigma_3 \frac{1}{2}(z-z_0)}{(\sigma_1 z \sigma_1 z_0' - \sigma_3 z \sigma_3 z') \sigma \frac{1}{2}(z+z') \sigma \frac{1}{2}(z-z_0') \sigma_2 \frac{1}{2}(z+z_0') \sigma_2 \frac{1}{2}(z-z_0')}, \quad (35)$$

17. It may be noticed that in the notation of Jacobi's elliptic functions

$$Z = \frac{\mathrm{dn}^2 z}{\mathrm{cn}\, z}, \quad k = \frac{1}{\sqrt{2}}. \qquad (35a)$$

So that

$$\Psi + \iota\Phi = \frac{m}{2\pi} \log \left\{ \left(\frac{\frac{1}{2}(1 + \mathrm{cn}^2 z)}{\mathrm{cn}\, z} - \frac{\frac{1}{2}(1 + \mathrm{cn}^2 z_0)}{\mathrm{cn}\, z_0} \right) \Big/ \Big/ \left(\frac{\frac{1}{2}(1 + \mathrm{cn}^2 z)}{\mathrm{cn}\, z} - \frac{\frac{1}{2}(1 + \mathrm{cn}^2 z_0')}{\mathrm{cn}\, z_0'} \right) \right\},$$

or

$$\Psi + \iota\Phi = \frac{m}{2\pi} \log \frac{(\mathrm{cn}\, z - \mathrm{cn}\, z_0)(1 - \mathrm{cn}\, z\, \mathrm{cn}\, z_0)}{(\mathrm{cn}\, z - \mathrm{cn}\, z_0')(1 - \mathrm{cn}\, z\, \mathrm{cn}\, z_0')}, \qquad (36)$$

is the solution for a vortex at z_0.

As an example of the application of the method when the motion is due to sources or sinks, the solution for a source and sink each of strength m at the base angles marked 1 and -1 is

$$\Phi + \iota\Psi = \frac{m}{\pi} \log \frac{1 + \mathrm{cn}\, z}{1 - \mathrm{cn}\, z}. \qquad (37)$$

On the Steady Motion of an Annular Mass of Rotating Liquid.

By A. B. Basset, M. A., *Cambridge, England.*

1. The recent investigations of Poincaré[*] and Professor G. H. Darwin[†] have drawn attention to the problem of the figures of equilibrium of rotating masses of liquid; and in the present paper it is proposed to consider the steady motion of an annular mass of liquid whose cross-section is approximately circular, and which is rotating as a rigid body under the influence of its own attraction, about an axis through its centre of inertia which is perpendicular to the plane of its central line.

This problem has to some extent been dealt with by Poincaré, who has proved that such figures are possible forms of surfaces of equilibrium; but the subject is capable of further development, and the object of this paper is to show how a solution may be obtained to any degree of approximation by the aid of the Toroidal Function analysis which has been so successfully employed by Mr. W. M. Hicks[‡] in his investigations on circular vortex rings.

2. In employing toroidal functions in problems such as the present, I have found it convenient to make use of a modified form of the methods introduced by Mr. Hicks. This method, together with many of the formulae required, will be explained in Chapter XII of my *Treatise on Hydrodynamics*, but for the sake of completeness I shall proceed to give a preliminary sketch.

Writing $\bar{\omega} = (x^2 + y^2)^{\frac{1}{2}}$, and starting with the dipolar transformation

$$z + \iota\bar{\omega} = a \tan \frac{1}{2}(\xi + \iota\eta)$$

instead of the logarithmic form, and putting $C = \cosh\eta$, $c = \cos\xi$, $k = \varepsilon^{-\eta}$, it has been shown by Mr. Hicks that the potential of an anchor ring or tore, which is composed of a mass of matter of constant density ρ, may be expressed in the forms

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[*] Acta Mathematica, Vol. VII, p. 259.

[†] Phil. Trans., 1887, p. 379.

[‡] Phil. Trans., 1881, p. 609; 1884, p. 161; 1885, p. 725.

$$V = (C+c)^{\frac{1}{2}} \sum_0^{\infty} A_n P_n \cos n\xi$$

at an external point, and*

$$V' = -\frac{2}{3}\pi\rho r^2 + (C+c)^{\frac{1}{2}} \sum_0^{\infty} B_n Q_n \cos n\xi$$

where P_n and Q_n are the two zonal toroidal functions of degree n.

From the formulae given by Mr. Hicks† it appears that P_n and Q_n respectively contain the factors $2k^{-n+\frac{1}{2}}$ and $\pi k^{n+\frac{1}{2}}$, and we shall therefore find it convenient to write $2P_n k^{-n+\frac{1}{2}}$ and $\pi Q_n k^{n+\frac{1}{2}}$ for the functions which he denotes by P_n and Q_n. It will also be shown farther on that if $(C+c)^{\frac{1}{2}}$ be expanded in a series of cosines of multiples of ξ, the coefficient of $\cos n\xi$ will be a rational and integral function of k multiplied by $(2k)^{-\frac{1}{2}}$. We shall therefore write

$$V = (2b)^{\frac{1}{2}}(C+c)^{\frac{1}{2}} \sum A_n P_n (b/k)^{n-\frac{1}{2}} \cos n\xi \tag{1}$$

for the value of the potential at an external point, and

$$V' = -\frac{2}{3}\pi\rho r^2 + (2b)^{\frac{1}{2}}(C+c)^{\frac{1}{2}} \sum B_n Q_n (k/b)^{n+\frac{1}{2}} \cos n\xi \tag{2}$$

for its value at an internal point, where b is the value of k at the surface of the tore.

At the critical circle $\eta = \infty$, and therefore $k = 0$. Now throughout the whole of the present paper the cross-section of the tore will be supposed to be small in comparison with its aperture, and as we shall only require to consider the values of the quantities in the neighbourhood of the tore, the following approximate values of P and Q and their differential coefficients with respect to k, which are denoted by accents, will be sufficient for our purpose, viz:

$$\left.\begin{array}{lll} P_0 = L + \frac{1}{4}k^2(L-1), & P_0' = -k^{-1} + \frac{1}{2}\ \left(L - \frac{3}{2}\right), & \\ & & P_0'' = k^{-2} + \frac{1}{2}\left(L - \frac{5}{2}\right), \\ P_1 = 1 + \frac{1}{2}k^2\left(L - \frac{1}{2}\right), & P_1' = \frac{1}{2}k\left(L - \frac{3}{2}\right), & P_1'' = \frac{1}{2}\left(L - \frac{5}{2}\right), \\ P_2 = \frac{2}{3}\left(1 + \frac{3}{4}k^2\right), & P_2' = k, & P_2'' = 1, \end{array}\right\} \tag{3}$$

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* ρ is supposed to be expressed in astronomical units.

† Phil. Trans., 1884, equations (9) and (10).

where $L = \log 4/k$; also

$$\left.\begin{aligned} Q_0 &= 1 + \frac{1}{4}k^2, & Q_0' &= \frac{1}{2}k, & Q_0'' &= \frac{1}{2}, \\ Q_1 &= \frac{1}{2}\left(1 + \frac{3}{8}k^2\right), & Q_1' &= \frac{3}{8}k, & Q_1'' &= \frac{3}{8}, \\ Q_2 &= \frac{3}{8}\left(1 + \frac{5}{12}k^2\right), & Q_2' &= \frac{5}{16}k, & Q_2'' &= \frac{5}{16}, \\ Q_3 &= \frac{5}{16} + \qquad\qquad , & &\text{etc.}, & &\text{etc.} \end{aligned}\right\} \qquad (4)$$

3. We shall also require the expansion of r^2, $\bar{\omega}^2$, and $(C+c)^{\frac{1}{2}}$.

Expansion of r^2.

$$\begin{aligned} \frac{r^2}{a^2} = \frac{C-c}{C+c} &= \frac{1-2kc+k^2}{1+2kc+k^2} \\ &= 1 - 4kc + 8k^2c^2 + 4k^3(c-4c^3) - 16k^4(c^2-2c^4), \end{aligned}$$

higher powers than k^4 being neglected; whence

$$\begin{aligned} r^2 = a^2\{1 + 4k^2 + 4k^4 - 4(k+2k^3)\cos\xi + 4(k^2+2k^4)\cos 2\xi \\ - 4k^3\cos 3\xi + 4k^4\cos 4\xi\}. \qquad (5) \end{aligned}$$

Expansion of $\bar{\omega}^2$.

$$\begin{aligned} \frac{\bar{\omega}^2}{a^2} &= \frac{(1-k^2)^2}{(1+2kc+k^2)^2} \\ &= 1 - 4kc + 4k^2(3c^2-1) + 4k^3(5c-8c^3) + 8k^4(1-9c^2+10c^4), \end{aligned}$$

whence

$$\begin{aligned} \bar{\omega}^2 = a^2\{1 + 2k^2 + 2k^4 - 4(k+k^3)\cos\xi + 2(3k^2+2k^4)\cos 2\xi \\ - 8k^3\cos 3\xi + 10k^4\cos 4\xi. \qquad (6) \end{aligned}$$

Expansion of $(C+c)^{\frac{1}{2}}$.

$$\begin{aligned} (2k)^{\frac{1}{2}}(C+c)^{\frac{1}{2}} &= (1+2kc+k^2)^{\frac{1}{2}} \\ &= 1 + kc + \frac{1}{2}k^2(1-c^2) - \frac{1}{2}k^3(c-c^3) - \frac{1}{4}k^4\left(\frac{1}{2} - 3c^2 + \frac{5}{2}c^4\right), \end{aligned}$$

whence

$$\begin{aligned} (C+c)^{\frac{1}{2}} = (2k)^{-\frac{1}{2}}\Big\{1 + \frac{1}{4}k^2 + \frac{1}{64}k^4 + \left(k - \frac{1}{8}k^3\right)\cos\xi \\ - \frac{1}{4}k^2\left(1 - \frac{1}{4}k^2\right)\cos 2\xi + \frac{1}{8}k^3\cos 3\xi - \frac{5}{64}k^4\cos 4\xi\Big\}. \qquad (7) \end{aligned}$$

4. Having obtained these preliminary results, we are in a position to consider the steady motion of the tore.

If Ω be the angular velocity in steady motion, the condition to be satisfied at the surface of the tore is

$$V + \frac{1}{2}\Omega^2(x^2+y^2) = \text{const.} \tag{8}$$

We are not, however, at liberty to assume that the cross-section is an exact circle in steady motion, and we shall therefore suppose that its equation is

$$\varepsilon^{-\eta} = k = b(1+\beta_1\cos\xi + \beta_2\cos 2\xi + \ldots) \tag{9}$$

where b is the mean value of k at the surface, which is supposed to be small in comparison with a, the radius of the critical circle; and we shall also assume that β_n is a small quantity of the order b^n. We must therefore first find the potential of an annular mass of matter of uniform density ρ whose cross-section is determined by (9). This is effected by assuming that the potentials at an external and internal point are respectively given by (1) and (2), and determining the coefficients from the consideration that at the surface the values of V and V' must differ by a constant, and that the values of dV/dk and dV'/dk must be equal. We shall thus obtain the values of the coefficients in terms of the β's. The resulting value of V and the surface value of $\bar{\omega}^2$ must then be substituted in (8), and the coefficients of the cosines of multiples of ξ equated to zero. This will determine the values of the β's.

The preceding method will enable us to determine the values of the β's to any degree of approximation that may be desired, but in order to avoid unnecessary complication, the investigation will be confined to the determination of the first term of β_1, which is sufficient to prove the existence of annular figures of equilibrium. From the course of the work it is evident that a higher approximation could be obtained with some additional labour.

We shall find that B_n is of the order b^n, but that A_n is of the order b^{n+2}, and we shall commence with the determination of V' and dV'/dk. In calculating the surface value of the former quantity, it will be unnecessary to proceed farther than the term involving $\cos 2\xi$, or to include terms of a higher order than the second; but in calculating dV'/dk it will be necessary, previously to performing the differentiation, to retain the term $\cos 3\xi$ together with quantities of the third order in the coefficients, since the terms of the third order reduce upon differentiation to terms of the second order.

5. *Calculation of V'.*

From (2), (5) and (7) we obtain

$$V' = -\frac{2}{3}\pi\rho a^2\{1+4k^2-4(k+2k^3)\cos\xi + 4k^2\cos 2\xi - 4k^3\cos 3\xi\} + GH', \tag{10}$$

where

$$G = 1 + \frac{1}{4}k^2 + \left(k - \frac{1}{8}k^3\right)\cos\xi . - \frac{1}{4}k^2\cos 2\xi + \frac{1}{8}k^3\cos 3\xi, \tag{11}$$

$$H' = B_0Q_0 + B_1Q_1(k/b)\cos\xi + B_2Q_2(k/b)^2\cos 2\xi + B_3Q_3(k/b)^3\cos 3\xi. \tag{12}$$

Omitting terms of a higher order than the second, the surface values of the quantities are

$$-\frac{2}{3}\pi\rho r^2 = -\frac{2}{3}\pi\rho a^2\{1 - 2b\beta_1 + 4b^2 - 4b\cos\xi + (4b^2 - 2b\beta_1)\cos 2\xi\},$$

$$G = 1 + \frac{1}{4}b^2 + \frac{1}{2}b\beta_1 + b\cos\xi + \frac{1}{2}\left(b\beta_1 - \frac{1}{2}b^2\right)\cos 2\xi, \tag{13}$$

$$H' = B_0\left(1 + \frac{1}{4}b^2\right) + \frac{1}{4}B_1\beta_1 + \frac{1}{2}B_1\cos\xi + \left(\frac{1}{4}B_1\beta_1 + \frac{3}{8}B_2\right)\cos 2\xi.$$

Therefore the surface value of V' is

$$V' = \text{const} + \left(\frac{8}{3}\pi\rho a^2 b + B_0 b + \frac{1}{2}B_1\right)\cos\xi$$
$$+ \left\{-\frac{2}{3}\pi\rho a^2(4b^2 - 2b\beta_1) + \frac{1}{4}B_1\beta_1 + \frac{1}{4}B_1 b + \frac{3}{8}B_2\right\}\cos 2\xi. \tag{14}$$

6. *Calculation of dV'/dk.*

From (10), (11) and (12) we obtain

$$\frac{dV'}{dk} = -\frac{8}{3}\pi\rho a^2\{2k - (1 + 6k^2)\cos\xi + 2k\cos 2\xi - 3k^2\cos 3\xi\}$$
$$+ H'\frac{dG}{dk} + G\frac{dH'}{dk}. \tag{15}$$

Now $$\frac{dG}{dk} = \frac{1}{2}k + \left(1 - \frac{3}{8}k^2\right)\cos\xi - \frac{1}{2}k\cos 2\xi + \frac{3}{8}k^2\cos 3\xi. \tag{16}$$

Also

$$H' = B_0\left(1 + \frac{1}{4}k^2\right) + \frac{1}{2}B_1\left(1 + \frac{3}{8}k^2\right)(k/b)\cos\xi + \frac{3}{8}B_2(k/b)^2\cos 2\xi$$
$$+ \frac{5}{16}B_3(k/b)^3\cos 3\xi.$$

Therefore

$$H'\frac{dG}{dk} = \frac{1}{2}B_0k + \frac{1}{4}B_1k/b + \left\{B_0\left(1 - \frac{1}{8}k^2\right) + \frac{1}{8}B_1k^2/b\right.$$
$$\left. + \frac{3}{16}B_2(k/b)^2\right\}\cos\xi + \left(-\frac{1}{2}B_0k + \frac{1}{4}B_1k/b\right)\cos 2\xi, \tag{17}$$

the term involving cos 3ξ being omitted, as it is not required. Also

$$\frac{dH'}{dk}=\frac{1}{2}B_0k+\frac{1}{2}B_1b^{-1}\left(1+\frac{9}{8}k^2\right)\cos\xi+\frac{3}{4}B_2b^{-2}k\cos 2\xi$$
$$+\frac{15}{16}B_3b^{-3}k^2\cos 3\xi,$$

therefore

$$G\frac{dH'}{dk}=\frac{1}{2}B_0k+\frac{1}{4}B_1k/b+\left\{\frac{1}{2}B_1\left(b^{-1}+\frac{5}{4}k^2/b\right)+\frac{1}{2}B_0k^2\right.$$
$$\left.+\frac{3}{8}B_2k^2/b^2\right\}\cos\xi+\left(\frac{1}{4}B_1k/b+\frac{3}{4}B_2k/b^2\right)\cos 2\xi. \tag{18}$$

Substituting from (17) and (18) in (15), we obtain

$$\frac{dV'}{dk}=-\frac{16}{3}\pi\rho a^2k+B_0k+\frac{1}{2}B_1k/b$$
$$+\left\{\frac{8}{3}\pi\rho a^2(1+6k^2)+B_0\left(1+\frac{3}{8}k^2\right)+\frac{1}{2}B_1\left(b^{-1}+\frac{3}{2}k^2/b\right)+\frac{9}{16}B_2(k/b)^2\right\}\cos\xi$$
$$+\left\{-\frac{16}{3}\pi\rho a^2k-\frac{1}{2}B_0k+\frac{1}{2}B_1k/b+\frac{3}{4}B_2k/b^2\right\}\cos 2\xi. \tag{19}$$

Putting for k its value from (9), the surface value is

$$\frac{dV'}{dk}=\left(-\frac{16}{3}\pi\rho a^2b+B_0b+\frac{1}{2}B_1\right)(1+\beta_1\cos\xi)$$
$$+\left\{\frac{8}{3}\pi\rho a^2(1+6b^2)+B_0\left(1+\frac{3}{8}b^2\right)+\frac{1}{2}B_1b^{-1}\left(1+\frac{3}{2}b^2\right)+\frac{9}{16}B_2\right\}\cos\xi$$
$$+\left\{-\frac{16}{3}\pi\rho a^2b-\frac{1}{2}B_0b+\frac{1}{2}B_1+\frac{3}{4}B_2/b\right\}\left(\frac{1}{2}\beta_1\cos\xi+\cos 2\xi\right). \tag{20}$$

From (19) or (20) it appears that the most important terms of dV'/dk are of zero order; and when we have calculated the value of dV/dk, it will be found that this circumstance requires that A_n should be of the order b^{n+2}.

7. *Calculation of V.*

Putting

$$H=A_0P_0+A_1(b/k)P_1\cos\xi+A_2(b/k)^2P_2\cos 2\xi, \tag{21}$$

we have $\quad V=GH.$

The surface value of G is given by (13); also by (3) at the surface,

$$P_0(k) = P_0(b) + \frac{1}{4}\beta_1^2 - \beta_1 \cos\xi + \left(\frac{1}{4}\beta_1^2 - \beta_2\right)\cos 2\xi,$$

$$(b/k)P_1 \cos\xi = \cos\xi - \frac{1}{2}\beta_1 - \frac{1}{2}\beta_1 \cos 2\xi,$$

$$(b/k)^2 P_2 \cos 2\xi = \frac{2}{3}\cos 2\xi.$$

The surface value of H is therefore

$$H = A_0\left\{L + \frac{1}{4}b^2(L-1) + \frac{1}{4}\beta_1^2\right\} - \frac{1}{2}A_1\beta_1 - (A_0\beta_1 - A_1)\cos\xi$$
$$+ \left\{A_0\left(\frac{1}{4}\beta_1^2 - \beta_2\right) - \frac{1}{2}A_1\beta_1 + \frac{2}{3}A_2\right\}\cos 2\xi. \tag{22}$$

Hence the surface value of V is

$$V = \text{const} + \{A_0(Lb - \beta_1) + A_1\}\cos\xi$$
$$+ \left\{A_0\left(\frac{1}{4}\beta_1^2 - \beta_2 - \frac{1}{2}b\beta_1\right) + \frac{1}{2}A_1(b - \beta_1) + \frac{2}{3}A_2\right\}\cos 2\xi. \tag{23}$$

8. *Calculation of dV/dk.*

We have $$\frac{dV}{dk} = G\frac{dH}{dk} + H\frac{dG}{dk}.$$

Now from (21) and (3),

$$\frac{dH}{dk} = A_0\left\{-k^{-1} + \frac{1}{2}k\left(L - \frac{3}{2}\right)\right\} - A_1 b k^{-2}\left\{1 + \frac{1}{2}k^2\left(L - \frac{1}{2}\right)\right\}\cos\xi$$
$$+ \frac{1}{2}A_1 b\left(L - \frac{3}{2}\right)\cos\xi + \text{etc.}$$

This has to be multiplied by G whose term of lowest order is unity. Now the constant terms in dV'/dk are of the first order, and therefore the constant terms in dV/dk must also be of the same order, whence A_0/b must be of the first order, and therefore A_0 must be of the second order of small quantities. Similarly it can be shown that all the other A's are of an order two degrees higher than their indices. We can therefore considerably simplify the calculation, since we do not require to retain quantities of a higher order than the second. We thus obtain

$$G\frac{dH}{dk} = -A_0/k - (A_1 b/k^2 + A_0)\cos\xi;$$

also from (16),

$$H\frac{dG}{dk} = A_0 P_0 \cos\xi = A_0 L \cos\xi,$$

whence
$$\frac{dV}{dk} = -A_0/k + (A_0L - A_0 - A_1b/k^2)\cos\xi,$$
and the value of this at the surface is
$$\frac{dV}{dk} = -A_0/b + (A_0\beta_1/b + A_0L - A_0 - A_1/b)\cos\xi; \tag{24}$$
the term involving cos 2ξ being of the third order, is omitted.

9. We can now obtain the value of V in terms of the β's. From (23) it appears that the coefficients of cos ξ and cos 2ξ in V are of the third and fourth orders respectively; and since these quantities are equal to the coefficients of the corresponding terms in V', it follows from (14) that to the lowest order
$$0 = \frac{8}{3}\pi\rho a^2b + B_0b + \frac{1}{2}B_1, \tag{25}$$
$$0 = -\frac{2}{3}\pi\rho a^2(4b^2 - 2b\beta_1) + \frac{1}{4}B_1(b + \beta_1) + \frac{3}{8}B_2. \tag{26}$$

Equating the constant terms in dV/dk and dV'/dk, we obtain
$$-\frac{16}{3}\pi\rho a^2b + B_0b + \frac{1}{2}B_1 = -A_0/b,$$
whence by (25)
$$A_0 = \frac{8}{3}\pi\rho a^2b^2. \tag{27}$$

Since the coefficient of cos 2ξ in dV/dk is of the third order, it follows from (20) that
$$-\frac{16}{3}\pi\rho a^2b^2 - \frac{1}{2}B_0b^2 + \frac{1}{2}B_1b + \frac{3}{4}B_2 = 0. \tag{28}$$

Equating coefficients of cos ξ in dV/dk and dV'/dk and using (25) and (28), we obtain
$$\begin{aligned} A_0(\beta_1/b + L - 1) - A_1/b &= -8\pi\rho a^2b\beta_1 + 16\pi\rho a^2b^2 + \frac{3}{8}B_0b^2 + \frac{3}{4}B_1b + \frac{9}{16}B_2 \\ &= -8\pi\rho a^2b\beta_1 + 18\pi\rho a^2b \end{aligned}$$
by (28), whence
$$A_1 = \frac{2}{3}\pi\rho a^2b^2(16\beta_1 + 4Lb - 31b). \tag{29}$$

Again from (6),
$$\frac{1}{2}\Omega^2\varpi^2 = \frac{1}{2}\Omega^2a^2(1 - 4b\cos\xi + \ldots.).$$

Substituting this together with the value of V from (23) in (8), and equating the coefficient of cos ξ to zero, we obtain
$$\beta_1 = \frac{1}{12}b(31 + 3\Omega^2/\pi\rho b^2 - 8L), \tag{30}$$
which determines β_1.

10. If we take the radius of the critical circle as the unit of length, it might be thought that if we gave to b any small numerical value, the value of Ω would be arbitrary, subject only to the condition that the resulting value of β_1 is a small numerical quantity of the same order as b. This, however, is not the case, for we have tacitly assumed that the pressure does not become negative at any point in the interior of the ring. If the pressure did become negative at any internal point, this would indicate the existence of a hollow space within the ring, and the preceding investigation would be no longer applicable, for the potential at any point in the substance of a ring which contained a hollow would involve the P functions as well as the Q functions; and we must therefore find the condition that the pressure should not become negative within the ring.

Since V' is the attraction potential, the pressure p is determined by the equation

$$p/\rho = V' + \frac{1}{2}\Omega^2(x^2 + y^2) + C.$$

Let P be the pressure along the critical circle where $k = 0$, then from (10), (11) and (12),

$$P/\rho = -\frac{2}{3}\pi\rho a^2 + B_0 + \frac{1}{2}\Omega^2 a^2 + C.$$

We have already found the conditions that the coefficients of $\cos\xi$, etc., should vanish in p, hence to determine the condition that p should vanish at the free surface, all that we require is the constant term in V'. By §6, this is

$$-\frac{2}{3}\pi\rho a^2(1 + 4b - 2b\beta_1) + B_0 + \frac{1}{2}b\left(B_0 b + \frac{1}{2}B_1\right) + \frac{1}{2}\beta_1\left(B_0 b + \frac{1}{2}B_1\right)$$
$$= -\frac{2}{3}\pi\rho a^2(1 + 6b^2) + B_0$$

by (25); whence

$$0 = -\frac{2}{3}\pi\rho a^2(1 + 6b^2) + B_0 + \frac{1}{2}\Omega^2 a^2(1 + 2b^2 - 2b\beta_1) + C$$

and $P/\rho = 4\pi\rho a^2 b^2 - \Omega^2 a^2(b^2 - b\beta_1)$.
We must therefore have

$$4\pi\rho - \Omega^2 + \Omega^2\beta_1/b > 0. \tag{31}$$

Let $\lambda = \Omega^2/4\pi\rho$, then substituting the value of β_1 from (30), the condition becomes

$$\lambda^2 - \left(\frac{2}{3}L - \frac{19}{12}\right)b^2\lambda + b^2 > 0. \tag{32}$$

11. As a numerical example, let $b = .1$, then

$$\beta_1 = .0124 + 10\lambda,$$

and if we put $\lambda = .01$, the left-hand side of (32) becomes .9224, and is therefore positive, and therefore

$$\beta_1 = .1124,$$
$$\Omega^2/4\pi\rho = .01$$

are solutions of the problem.

Since (30) may be written in the form

$$\beta_1 = \frac{1}{b}\left(\frac{31}{12}b^3 + \lambda - \frac{2}{3}b^2L\right),$$

it follows that if b is very small, β_1 cannot be of the same order unless λ is very small.

It is also necessary to point out that the preceding analysis proceeds on the assumption that β_2 is of the order b^2, and it is quite possible that if a series of values were assigned to b and λ which satisfy (30) and (32), these values might not satisfy the condition that β_2 should be of the order b^2, or the inequality corresponding to (32) which would be obtained by carrying the approximation one degree higher. This difficulty would not be avoided by calculating the value of β_2, since the same difficulty would exist with regard to β_3. It therefore appears that although toroidal function analysis throws some light on the solution of the problem, it fails to give a perfectly satisfactory approximate solution. The preceding results indicate that for small angular velocities an annular figure exists whose cross-section is approximately circular, but for large angular velocities the cross-section would probably become highly elliptical, and the ring would become flattened; and that if this quantity increased beyond a certain limit, the ring would break up.

July 30, 1888.

Die Begriffe Gruppe und Invariante.[*]

Von Sophus Lie.

In einer Abhandlung von Sylvester im *American Journal of Mathematics*, Bd. 9, No. 2, findet sich ein Brief von Halphen, Mitglied der Akademie der Wissenschaften in Paris. In diesem Briefe betrachtet Halphen eine Schaar von algebraischen Transformationen

$$(1) \qquad x'_\varkappa = f_\varkappa(x_1 \dots . x_n, a_1 \dots . a_r) \qquad (\varkappa = 1 \dots . n)$$

mit r Parametern $a_1 \dots . a_r$ zwischen den Veränderlichen $x_1 \dots . x_n$ und $x'_1 \dots . x'_n$, stellt die Frage nach allgemeinen Kriterien dafür, ob eine solche Schaar von Transformationen Invarianten besitzt oder nicht und verspricht dieses noch nicht in voller Allgemeinheit behandelte Problem vollständig zu erledigen.[†]

Soviel ich sehe ist das ihm doch nicht gelungen. Im Laufe seines Briefes macht er nämlich verschiedene Annahmen, die wirkliche Beschränkungen des Problems nach sich ziehen. Und selbst das hiermit beschränkte Problem hat er, wie mir scheint, nicht vollständig erledigt. Die von ihm formulirten Kriterien sind zwar hinreichend aber nicht nothwendig.[‡]

1. Zunächst schicke ich einige Bemerkungen voraus über die eigenthümliche Bedeutung, in welcher Halphen das Wort *Gruppe* braucht.

Halphen sagt, dass die Gleichungen (1) eine Gruppe bestimmen, wenn sich aus den beiden Gleichungssystemen

$$\begin{aligned} x'_\varkappa &= f_\varkappa(x_1 \dots . x_n \;\; a_1 \dots . a_r) \\ x''_\varkappa &= f_\varkappa(x_1 \dots . x_n \;\; b_1 \dots . b_r) \end{aligned} \qquad (\varkappa = 1 \dots . n)$$

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[*] Reprinted from the Berichte der K. Sächs. Gesellschaft der Wissenschaften, Mathematisch-Physische Classe, Sitzung am 1. August 1887.

[†] Halphen drückt sich so aus: "*Dans des théories diverses on a rencontré des Invariants sans qu'on ait pénétré la cause générale de leur existence. C'est cette lacune qu'il s'agit ici de faire disparaître.*"

[‡] Es scheint Halphen unbekannt zu sein, dass ich das im Texte besprochene Problem für den Falle dass die Schaar der vorgelegten Transformationen (1) von infinitesimalen Transformationen erzeugt ist (oder sich erzeugen lässt) vollständig erledigt habe. Meine alten Untersuchungen über diesen und verwandte Gegenstände sind resumirt in den Math. Ann. Bd. XXIV, 1884.

durch Elimination der Grössen x_κ Relationen von der Form

$$x''_\kappa = f_\kappa (x'_1 \dots . x'_n , A_1 \dots . A_r)$$

ableiten lassen. Dabei macht er nicht wie ich in meiner Note in den Göttinger Nachrichten 1874 und in vielen späteren Arbeiten die Annahme, dass die Grössen A_κ nur von den a_i und b_j abhängen, sondern er lässt sie ganz beliebige Grössen bedeuten. Diese seine Definition des Begriffes Gruppe scheint mir indess nicht naturgemäss.

Es unterliegt ja keinem Zweifel, dass der Begriff Transformationsgruppe sich allgemeiner fassen lässt, als in meiner oben citirten Note geschehen. In der That habe ich schon im Jahre 1871 (Verhandlungen der Gesellschaft der Wissenschaften zu Christiania, S. 243) die folgende allgemeine Definition aufgestellt: *Eine Schaar von Transformationen bildet eine Gruppe, wenn zwei Transformationen, nach einander ausgeführt, immer eine Transformation der Schaar ergeben.* Diese Definition giebt, scheint es mir, die richtige und allgemeine Uebertragung des Gruppenbegriffes der Substitutionentheorie auf die Transformationstheorie. Unter den Transformationsgruppen giebt es aber mehrere verschiedene Kategorien, für die ich besondere Bezeichnungen benutze. Insbesondere bezeichne ich eine Gruppe dann als *endlich* und *continuirlich*, wenn ihre Transformationen durch *ein* Gleichungssystem mit r Parametern:

$$x'_\kappa = f_\kappa (x_1 \dots . x_n, a_1 \dots . a_r)$$

bestimmt werden.

Was Halphen's Definition des Begriffes Gruppe angeht, so möchte ich fast glauben, dass dieselbe nicht nach dem Wortlaute zu verstehen ist. Vielleicht sind einige Voraussetzungen nicht mit angegeben, z. B. dass sich zwischen den Gleichungen (1) die Parameter a_κ eliminiren lassen sollen. Es scheint mir nämlich undenkbar, dass Halphen jedes System Transformationsgleichungen (1), aus welchen sich die Parameter nicht eliminiren lassen, als eine Gruppe bezeichnen will, und doch ist er nach dem Wortlaute seiner Definition dazu gezwungen. Will Halphen auf der anderen Seite zu seiner Definition des Begriffes Gruppe etwa die Forderung hinzufügen, dass sich die Parameter eliminiren lassen sollen, so folgt z. B., dass die Schaar aller projectiven Transformationen eines Raumes keine Gruppe von Punkttransformationen dieses Raumes bildet.

Es scheint wünschenswerth, dass Halphen den Sinn seiner Definition präcisirt, wenn er sich nicht der von mir benutzten Terminologie anschliessen kann.

Nehme ich seine Definition nach dem Wortlaute und nenne ich ferner eine Function $\phi(x_1 \dots . x_n)$ dann eine Invariante der Transformationen (1), wenn die Identität

$$\phi(x_1' \dots . x_n') = \phi(x_1 \dots . x_n)$$

besteht, so ist sein Hauptsatz: *Les invariants sont l'apanage exclusif des substitutions formant groupe* unrichtig. Das werden uns später zwei Beispiele zeigen.

2. Indem Halphen das von ihm gestellte Problem in Angriff nimmt, macht er erstens und zwar ausdrücklich die Annahme, dass die Zahl. n der Veränderlichen grösser ist, als die Zahl r der Parameter. Da aber schon das Beispiel

$$x_1' = x_1, \quad x_2' = \frac{x_2 + a_1}{a_2 x_2 + a_3}$$

zeigt, dass Invarianten sehr gut vorkommen können, wenn $r \geqq n$ ist, so ist seine Annahme eine Beschränkung des Problems Er macht noch zwei weitere Annahmen, nämlich, dass die Elimination der Parameter gerade $n - r$ Relationen zwischen den x_κ und x_κ' liefert und dass die Gleichungen (1), wenn sie Invarianten besitzen, deren gerade $n - r$ haben.

Dass auch diese letzten Annahmen wesentliche Beschränkungen des Problems sind, liesse sich ebenfalls leicht an Beispielen zeigen. Darauf gehe ich jedoch hier nicht ein, sondern wende mich zu dem von Halphen behandelten *reducirten* Probleme.

3. Halphen behauptet, dass ein System von algebraischen Transformationsgleichungen

$$(1') \qquad x_\kappa' = f_\kappa(x_1 \dots . x_n, \; a_1 \dots . a_r), \qquad n > r,$$

aus welchem sich durch Elimination der Parameter gerade $n - r$ Relationen zwischen den x_κ und x_κ' ergeben, (dann und) nur dann gerade $n - r$ unabhängige Invarianten besitzt, wenn es in seinem Sinne des Wortes eine Gruppe bestimmt.

Aber diese Regel stimmt doch, wie ich sie verstehe, nicht mit dem Beispiele:

$$(2) \qquad x_1' = x_1 + a, \quad x_2' = -x_2.$$

Hier ist $n = 2$, $r = 1$, also $n > r$; die Elimination des Parameters a giebt $n - r = 1$ von a freie Relation; die Gleichungen (2) haben andererseits gerade $n - r = 1$ unabhängige Invariante, nämlich x_2^2. Nach Halphen's Regel müssten also die Gleichungen (2) in seinem Sinne des Wortes eine Gruppe bilden. Aber

aus $\qquad x_1' = x_1 + a, \quad x_2' = -x_2$

und $\qquad x_1'' = x_1 + b, \quad x_2'' = -x_2$

folgt $\qquad x_1'' = x_1' + b - a, \quad x_2'' = x_2'$

und diese Gleichungen haben die Form (2) nicht.

Betrachten wir andererseits das Beispiel

$$x_1' = x_1 + 1, \quad x_2' = x_2 + a. \tag{3}$$

Hier ist wiederum $n = 2$, $r = 1$. Die Elimination des Parameters a giebt wiederum $n - r = 1$ Relation zwischen den x und x'. Andererseits haben die Transformationsgleichungen (3) gerade $n - r = 1$ unabhängige Invariante nämlich $\operatorname{tg}(\pi x_1)$. Est ist ja

$$\operatorname{tg}(\pi x_1') = \operatorname{tg}(\pi x_1 + \pi) = \operatorname{tg}(\pi x_1).$$

Unsere Gleichungen bilden aber keine Gruppe in Halphen's Sinne des Wortes; denn aus

$$x_1' = x_1 + 1, \quad x_2' = x_2 + a$$

und

$$x_1'' = x_1 + 1, \quad x_2'' = x_2 + b$$

folgen die Gleichungen

$$x_1'' = x_1', \qquad x_2'' = x_2' + b - a,$$

welche nicht die Form (3) haben.

4. Will man das von Halphen gestellte Problem in voller Allgemeinheit behandeln, so dürfte es zweckmässig sein, die Ausdrucksweise der Mannigfaltigkeitslehre zu benutzen, also $x_1 \dots x_n$ als Punktcoordinaten eines n-fachen Raumes zu betrachten. Hier werde ich jedoch die Beschränkung einführen, dass die Transformationen

$$x_\kappa' = f_\kappa(x_1 \dots x_n,\ a_1 \dots a_r) \tag{1}$$

Cremona'sche Transformationen sein sollen.

Bei jeder Transformation einer derartigen Schaar geht ein Punkt x_κ von allgemeiner Lage in einen bestimmten neuen Punkt x_κ' über. Dieser neue Punkt hängt von den Parametern a_κ ab. Der Inbegriff aller Punkte x_κ' bildet daher eine algebraische Mannigfaltigkeit, die höchstens r-fach ausgedehnt ist. Werden nun weiter auf einen beliebigen Punkt x_κ' noch einmal die Transformationen (1) ausgeführt, so nimmt derselbe neue Lagen x_κ'' an u. s. w.

Wir nehmen zu der Schaar der Transformationen (1) noch die Schaar der inversen Transformationen hinzu und denken uns jede von diesen Schaaren von Transformationen eine Reihe von Malen hintereinander ausgeführt. Dann können zwei verschiedene Fälle eintreten. Es ist denkbar, dass ein Punkt x_κ von allgemeiner Lage durch wiederholte Ausführung jener Transformationen in jeden Punkt von allgemeiner Lage im Raume übergeführt werden kann. Dann giebt es keine Invarianten.

Kann dagegen der Punkt x_κ nicht in alle Punkte des Raumes übergeführt werden, so bildet der Inbegriff aller Lagen, welche derselbe bei wiederholter Anwendung der betreffenden Transformationen annehmen kann, einen geometrischen Ort, der allerdings aus verschiedenen, ja sogar aus unendlich vielen irreducibeln algebraischen Mannigfaltigkeiten bestehen kann.

Jeder Punkt des Raumes gehört einem ganz bestimmten derartigen Orte an. Der Raum wird also in dieser Weise in unendlich viele Oerter zerlegt, wobei jeder Ort unter Umständen aus mehreren algebraischen Mannigfaltigkeiten besteht.

Lässt sich nun diese Zerlegung des Raumes in geometrische Oerter analytisch definiren durch die Gleichungen

$$\Omega_1 (x_1 \dots . x_n) = C_1 \dots . , \quad \Omega_m (x_1 \dots . x_n) = C_m,$$

in denen die C_k willkürliche Constanten bedeuten, so sind die $\Omega_\kappa (x_1 \dots . x_n)$ Invarianten der Schaar von Transformationen, und zwar erhält man so ein vollständiges System Invarianten. Es ist hierbei keineswegs sicher, dass die Ω_κ algebraische Functionen von den x sind, sie können sehr gut transcendente* Functionen sein. Die Ω_κ brauchen nach dem Vorangehenden nur so beschaffen zu sein, dass die Gleichungen (4) für jedes specielle Werthsystem der a_κ eine discrete, endlich oder unendliche Anzahl von algebraischen Mannigfaltigkeiten darstellen, deren Inbegriff bei den Transformationen (1) invariant bleibt.

Entsprechende Ueberlegungen geben zugleich alle invarianten Gleichungssysteme.

An einer anderen Stelle werde ich auf die hier skizzirten Theorien näher eingehen. Insbesondere werde ich Gruppen behandeln, die sich nicht durch *ein* Gleichungssystem, sondern erst durch mehrere Gleichungssysteme definiren lassen, von denen jedes Gleichungssystem gewisse willkürliche Parameter enthält und werde eine Invariantentheorie von derartigen Gruppen entwickeln. Das gelingt leicht, indem ich meine allgemeine Invariantentheorie der endlichen continuirlichen Gruppen mit der Invariantentheorie der discontinuirlichen Gruppen verbinde.

LEIPZIG, Juli 1887.

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* Es ist daher unrichtig, wenn Halphen behauptet, dass die Invarianten immer durch Elimination gefunden werden können. Dass die zu einer *continuirlichen endlichen Gruppe* gehörigen Invarianten, Differentialinvarianten, ja sogar alle zugehörigen invarianten Gleichungssysteme durch Elimination gefunden werden, habe ich längst bei vielen Gelegenheiten hervorgehoben. Und auch die Ausdehnung dieser Bemerkung auf eine jede vorgelegte Schaar von Transformationen $x'_\kappa = f_\kappa (x, a)$, die sich von infinitesimalen Transformationen erzeugen lässt, rührt von mir her.

Sur les formes quadratiques binaires à indéterminées conjuguées et les fonctions fuchsiennes.

PAR M. EMILE PICARD.

Considérons une forme quadratique binaire indéfinie à indéterminées conjuguées

$$axx_0 + bxy_0 + b_0x_0y + cyy_0, \quad bb_0 - ac > 0 \tag{1}$$

a et c étant des entiers reéls, b et b_0 deux entiers complexes conjugués ; x et y désignent deux indéterminées, ayant respectivement pour conjuguées x_0 et y_0. Soit une substitution quelconque

$$(x,\, y,\, Mx + Ny,\, Px + Qy)$$

à coefficients entiers complexes, et de déterminant *un*, transformant la forme précédente en elle-même. Le groupe des substitutions

$$\left(z,\, \frac{Mz + N}{Pz + Q}\right)$$

est un groupe fuchsien. J'ai, il y a quelques années (Annales de l'Ecole Normale, 1884), étudié à ce point de vue les formes quadratiques à indéterminées conjuguées, et j'ai montré notamment comment on peut construire un polygone fondamental du groupe et trouver ses substitutions fondamentales.

Je voudrais, dans cet article, compléter un peu cette étude, en portant particulièrement l'attention sur les substitutions *elliptiques* du groupe fuchsien ainsi engendré. De telles substitutions n'existeront d'ailleurs pas pour une forme (1) prise arbitrairement, et le groupe n'aura que des substitutions hyperboliques.

Une question intéressante se présentait dans cette étude ; c'est celle de la *réduction* des substitutions. J'ai employé pour cet objet les méthodes dont a fait usage M. Poincaré dans son beau mémoire sur les fonctions fuchsiennes et l'arithmétique (Journal de Jordan, 1887).

1. Considérons les substitutions binaires, à coefficients complexes,

$$\begin{matrix} x' = Mx + Ny \\ y' = Px + Qy \end{matrix}, \quad MQ - NP = 1,$$

transformant en elle-même la forme quadratique

$$axx_0 + bxy_0 + b_0x_0y + cyy_0, \qquad bb_0 - ac \neq 0.$$

Ces substitutions se partagent, comme il est bien connu, en substitutions elliptiques, hyperboliques et paraboliques. La nature de la substitution dépend de la somme

$$M + Q$$

qui est nécessairement réelle. Démontrons d'abord directement ce premier point: on a

$$\begin{aligned} aMM_0 + bMP_0 + b_0M_0P + cPP_0 &= a, \\ aMN_0 + bMQ_0 + b_0N_0P + cPQ_0 &= b, \\ aNN_0 + bNQ_0 + b_0N_0Q + cQQ_0 &= c \end{aligned}$$

et on tire de suite de ces équations

$$\begin{aligned} aM_0 + bP_0 &= aQ - b_0P, \\ b_0M_0 + cP_0 &= bM_0 - aN, \\ aN + b_0Q &= b_0Q_0 - cP_0, \\ b_0N_0 + cQ_0 &= cM - bN \end{aligned}$$

la seconde et la troisième, additionnées en croix, donnent

$$b_0(M_0 + Q_0) = b_0(M + Q).$$

Donc $M + Q = M_0 + Q_0$, c'est à dire que $M + Q$ est réelle.

Si on avait $b = 0$, la proposition n'en subsiste pas moins, car les équations précédentes donnent alors $Q = M_0$.

Les substitutions elliptiques correspondent au cas où

$$(M + Q)^2 = 0 \text{ ou } 1.$$

2. Nous allons maintenant chercher à effectuer la réduction des substitutions elliptiques. Inspirons nous pour cela des principes qui ont guidé M. Poincaré dans le mémoire déjà cité.

Soit une substitution $S = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ dont les coefficients a, b, c, d sont des entiers complexes, assujettis uniquement à vérifier la relation $ad - bc = 1$, et soit T une autre substitution de même forme.

La substitution S, et la transformée de S par T

$$T^{-1}ST$$

peuvent être regardées comme appartenant à la même *classe.*

Parmi toutes les substitutions d'une même classe, il y en a de particulièrement simples; ce sont ces substitutions *réduites* que nous allons chercher. On vérifie d'ailleurs immédiatement que deux substitutions de même classe ont les mêmes multiplicateurs, et, par suite, si une substitution est elliptique, toutes les substitutions de même classe sont elliptiques.

Cela posé, considérons la forme quadratique binaire

$$cx^2 + (d-a)xy - by^2$$

elle n'est pas altérée par la substitution

$$S = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

et si on applique à cette forme la substitution

$$T = \begin{vmatrix} \lambda & \mu \\ \lambda' & \mu' \end{vmatrix}$$

elle se transformera en une nouvelle forme

$$c'x^2 + (d'-a')xy - b'y^2$$

la substitution
$$S' = \begin{vmatrix} a' & b' \\ c' & d' \end{vmatrix}$$

n'étant autre chose que la transformée de S par T, c'est à dire $T^{-1}ST$. Donc, pour que deux substitutions S et S' soient de même classe, il faut et il suffit que les deux formes à coefficients complexes

$$cx^2 + (d-a)xy - by^2,$$
$$c'x^2 + (d'-a')xy - b'y^2$$

soient équivalentes.

3. Avant d'aller plus loin, il nous faut rappeler un théorème, relatif aux formes quadratiques binaires à coefficients et indéterminées complexes. On sait que Dirichlet a la premier étudié cette théorie dans un mémoire célèbre (Crelle, tome 24). Relativement à la réduction des formes

$$ax^2 + 2bxy + cy^2$$

a, b, c désignant des entiers complexes, le théorème fondamental est le suivant: on peut toujours trouver une forme équivalente à une forme donnée, pour laquelle on ait:

$$\frac{1}{2}\ \text{norme}\,(a) \geqslant \text{norme}\,(b),$$
$$\text{norme}\,(a) \leqslant \text{norme}\,(c);$$

on en conclut qu'en désignant par D l'invariant $b^2 - ac$, on a

$$\text{norme}\,(b) \leqslant \sqrt{\text{norme}\,D}$$

et par suite les normes de a, b, c sont limitées en fonction de la norme de D.

4. Ceci posé, revenons aux formes que nous avons rencontrées plus haut

$$F = cx^2 + (d-a)\,xy - by^2$$

le discriminant de $2F$ est

$$(d-a)^2 + 4bc \quad \text{ou} \quad (d+a)^2 - 4$$

les seuls cas qui nous intéressent sont ceux où

$$a + d = 0,\ \pm 1.$$

Soit d'abord $a + d = 0$; dans ce cas $d - a$ sera un nombre pair; la forme F a donc le type classique dans la théorie des formes binaires, et son discriminant est égal à -1.

Nous avons, par suite, à considérer les formes

$$ax^2 + 2bxy + cy^2$$

de discriminant -1.

Nous simplifierons beaucoup la discussion, en partant de cette remarque évidente que toute forme de discriminant -1 est équivalente à une forme dans laquelle $a = 0$, c'est à dire à une forme

$$\pm\, 2ixy + cy^2.$$

Tout d'abord en faisant la substitution

$$(x,\ y,\ x + \lambda y,\ y)$$

on voit que le coefficient c peut être ramené à une des quatre valeurs

$$0,\ 1,\ i,\ 1 + i.$$

D'autre part on peut se borner à prendre le signe +, car les autres formes sont équivalentes; on aura donc seulement les *quatre* formes

$$2ixy,\quad 2ixy + y^2,\quad 2ixy + iy^2,\quad 2ixy + (1+i)\,y^2$$

et ces quatre formes sont bien distinctes.

Nous avons donc les types de substitutions correspondants, en identifiant successivement

$$cx^2 + (d - a)\,xy - by^2$$

avec ces quatre formes.

Dans les quatre cas

$$d - a = 2i, \quad c = 0.$$

1° On aura

$$d - a = 2i \quad b = c = 0, \text{ et comme } ad = 1, \text{ on a } a(a + 2i) = 1$$

$$a^2 + 2ia - 1 = 0, \text{ on a } a = -i$$

donc substitution $\begin{vmatrix} -i & 0 \\ 0 & i \end{vmatrix}$

2°. $d - a = 2i, \quad c = 0, \quad b = -1$ donc $\begin{vmatrix} -i & -1 \\ 0 & +i \end{vmatrix}$

3°. $d - a = 2i, \quad c = 0, \quad b = -i$ donc $\begin{vmatrix} -i & -i \\ 0 & +i \end{vmatrix}$

4°. $d - a = 2i, \quad c = 0, \quad b = -(1 + i)$ donc $\begin{vmatrix} -i & -(1+i) \\ 0 & +i \end{vmatrix}$

5. Considérons maintenant le cas où $(a + d)^2 = 1$. Dans ce cas $d - a$ sera un nombre impair, et nous devrons considérer la forme

$$2cx^2 + 2(d - a)\,xy - 2by^2.$$

Nous avons donc à considérer des formes :

$$2Ax^2 + 2Bxy + 2Cy^2$$

de déterminant $B^2 - 4AG = -3$.

D'après la théorie de la réduction, on peut supposer que la norme de B est inférieure à $\sqrt{3}$. Nous aurons donc ici, pour une forme réduite

$$B = \pm 1$$

et, par suite $AG = 1$

d'où les seuls types :

$$2x^2 \pm 2xy + 2y^2 \text{ et } 2ix^2 \pm 2xy - 2iy^2,$$

mais $2x^2 + 2xy + 2y^2$ et $2x^2 - 2xy + 2y^2$ sont manifestement équivalents, et aussi

$$2ix^2 + 2xy - 2iy^2 \text{ et } 2ix^2 - 2xy - 2iy^2$$

puisqu'il suffira de faire sur la première forme la substitution

$$(x,\, y,\, x + iy,\, y)$$

pour retrouver la seconde.

Il nous reste donc

$$2x^2 + 2xy + 2y^2 \text{ et } 2ix^2 + 2xy - 2iy^2$$

qui ne sont pas équivalentes.

Ceci nous donne les quatre types de substitution:

$$\begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix}, \quad \begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix}, \quad \begin{vmatrix} -1 & i \\ i & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & i \\ i & 1 \end{vmatrix}.$$

6. Cherchons maintenant les formes qui se reproduisent pour les substitutions du trouvées au paragraphe (4).

Pour la première le type est

$$axx_0 + cyy_0,$$

pour la seconde on a les formes

$$2axx_0 + aixy_0 - aix_0y + cyy_0$$

la troisième substitution nous donne les formes

$$2axx_0 + axy_0 + ax_0y + cyy_0$$

et nous avons enfin pour la quatrième

$$2axx_0 + a(1 + i)\,xy_0 + a(1 - i)\,x_0y + cyy_0,$$

a et c représentant, dans ces diverses formes, des nombres entiers.

Ceci posé, si une forme donnée f admet des substitutions semblables,

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix}$$

pour lesquelles $M + Q = 0$, cette forme doit être équivalente à une forme rentrant dans un des quatre types précédents. Or les déterminants de ces formes sont respectivement

$$-ac, \; a^2 - 2ac, \; 2a^2 - 2ac$$

par suite pour une valeur donnée de ce déterminant, les formes sont en nombre limité; on aura donc seulement un nombre limité d'essais à faire, pour reconnaître si une forme donnée admet une substitution elliptique du type considéré.

7. Considérons maintenant le second ensemble de substitutions, celles du paragraphe (5). Prenons la première; les formes qui admettent cette substitution pour substitution semblable, rentrent dans le type

$$2axx_0 + (a + mi)\,xy_0 + (a - mi)\,x_0y + 2ayy_0. \tag{1}$$

Le discriminant de cette forme étant

$$m^2 - 3a^2$$

il pourra y avoir une infinité de ces formes ayant un discriminant donné.

Quant à la seconde substitution, elle n'est pas distincte de la première, puisqu'elle est égale à la substitution inverse, où on remplace x et y par $-x$ et $-y$.

Pareillement pour la troisième et la quatrième, nous avons le type de formes :

$$2axx_0 + (m + ai)\,xy_0 + (m - ai)\,x_0y + 2ayy_0 \tag{2}$$

et il y a encore ici une infinité de formes qui peuvent correspondre à un déterminant donné.

Si une forme donnée f admet des substitutions semblables

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix}$$

pour lesquelles $M + Q = \pm 1$, cette forme doit être équivalente à une forme (1) ou à une forme (2). Nous ne trouvons plus ici devant un nombre limité d'essais à tenter, comme dans le paragraphe précédent ; aussi, dans le cas actuel, sera-t-il préférable de ne pas rechercher directement les substitutions elliptiques de ce type.

8. Des deux paragraphes précédents, on conclut que, une forme f étant prise arbitrairement, *le groupe fuchsien auquel elle donne naissance n'admet pas de substitution elliptique.*

Quant aux substitutions paraboliques, nous avons déjà dit précédemment (mémoire cité), qu'il y en a seulement dans le cas où le discriminant de la forme est *une somme de deux carrés.*

9. Prenons, en particulier, la forme

$$xx_0 - Dyy_0$$

le groupe fuchsien correspondant admettra des substitutions elliptiques du premier type, mais il est aisé de voir qu'elle n'en admettra pas du second type. Soit en effet

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix}$$

une substitution transformant en elle-même la forme précédente, on aura nécessairement :

$$M_0 = Q,$$

donc

$$M + Q = M + M_0$$

or $M + M_0$, qui est un nombre pair, ne peut être égal à ± 1.

Un autre exemple, extrêmement simple, mais particulièrement intéressant, est celui de la forme

$$ixy_0 - ix_0y$$

toute substitution semblable de cette forme a ses coefficients réels, et le groupe correspondant n'est autre que le groupe modulaire

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix}$$

où M, N, P, Q sont quatre entiers réels, satisfaisant à l'unique condition:

$$MQ - NP = 1.$$

10. M. Poincaré, dans le mémoire que nous avons déjà cité, a donné une généralisation bien remarquable des équations modulaires. Les fonctions fuchsiennes, provenant d'une forme quadratique ternaire indéfinie jouissent en effet d'une propriété remarquable qu'on peut regarder comme la généralisation de la transformation des fonctions modulaires. Il va en être de même ici pour les fonctions fuchsiennes provenant d'une forme quadratique binaire à indéterminées conjuguées. Le théorème fondamental de M. Poincaré va subsister ici; on l'énoncera dans ce cas de la manière suivante: Soient f une forme quadratique binaire indéfinie à indéterminées conjuguées, et G le groupe fuchsien correspondant à cette forme. Considérons d'autre part une substitution transformant f en elle-même, mais dont les coefficients ne soient pas des entiers, mais seulement des nombres commensurables (réels ou complexes); à cette substitution correspond une substitution de la forme

$$\left(z, \frac{az+b}{cz+d}\right)$$

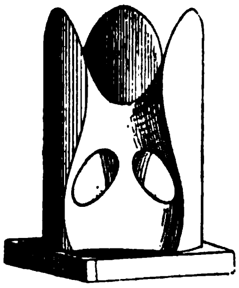
a, b, c, d étant des entiers, mais le déterminant $ad - bc$ étant alors différent de l'unité.

Ceci posé, si $F(z)$ désigne une fonction fuchsienne correspondant au groupe G, il y aura une relation algébrique entre

$$F(z) \text{ et } F\left(\frac{az+b}{cz+d}\right)$$

ce qui nous donne une nouvelle classe, très étendue, d'équations analogues aux équations modulaires.

TALLOIRES (HAUTE-SAVOIE), le 2 Septembre 1888.

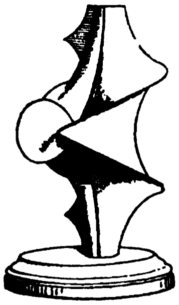


MODELS

MANUFACTURED BY

L. BRILL IN DARMSTADT.

(These Models are of plaster, unless otherwise stated.)



The models of Series I, II, V, VI, VIII, and for the most part of X and XIV, were constructed after the originals in the Mathematical Institute of the Royal Polytechnicum at Munich, under the direction of Prof. Dr. Brill, Dr. Klein and Dr. Dyck.

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CONTENTS.

PAGE

Remarque au sujet du théorème d'Euclide sur l'infinité du nombre des nombres premiers. Par M. JOSEPH PEROTT, ... 99

On the Theory of Groups. By Prof. CAYLEY, ... 139

Vortex Motion in certain Triangles. By A. E. H. LOVE, ... 158

On the Steady Motion of an Annular Mass of Rotating Liquid. By A. B. BASSET, . 172

Die Begriffe Gruppe und Invariante. Von SOPHUS LIE, ... 182

Sur les formes quadratiques binaires à indéterminées conjuguées et les fonctions fuchsiennes. Par M. EMILE PICARD, ... 187

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On the Construction of Intransitive Groups.

By Oskar Bolza.

If a substitution-group I is *intransitive*, the letters upon which it operates can be distributed into "*systems of intransitivity*,"

$$x_1, x_2 \dots ; \; y_1, y_2 \dots ; \; z_1, z_2 \dots ; \; \dots$$

such that the substitutions of I interchange among each other only the letters $x_1, x_2 \dots$; the letters $y_1, y_2 \dots$; the letters $z_1, z_2 \dots$, and so on, and connect transitively the letters of each system. Every substitution of I is then a product $g.h.k \dots$, g being a substitution between the letters x only, h a substitution between the y, and so on. The substitutions g which occur in the different substitutions of I form together a transitive group G, likewise the h form a group H, and so on.* We will agree, for shortness, to call these groups G, H, $K \dots$ the *transitive constituents* of the intransitive group I, and we propose to solve, in the present paper, the inverse problem:

Given the transitive substitution-groups G, H, $K \dots$ *between the letters* $x_1, x_2 \dots ; y_1, y_2 \dots ; z_1, z_2 \dots ; \dots$ *respectively, to find* ALL *the intransitive groups with the systems of intransitivity,* $x_1, x_2 \dots ; y_1, y_2 \dots ; z_1, z_2 \dots ; \dots$, *and whose transitive constituents are the groups* G, H, $K \dots$

There exists always *one self-evident* solution of this problem, viz. the group generated by multiplying by each other, in all possible ways, the substitutions of G, H, $K \dots$; but there *may* be still other solutions. For instance, if we choose for G the symmetrical group of three letters, for H the symmetrical group of three other letters, it is easily seen that we obtain intransitive groups with the transitive constituents G and H:

a) by multiplying every substitution of G by every substitution of H (order 36);

---

* Camille Jordan: Traité des substitutions, No. 44.

b) by multiplying every positive (gerade) substitution of G by every positive substitution of H, and every negative substitution of G by every negative substitution of H (order 18);

c) by multiplying every substitution of G by the same substitution of H, that is to say, 1 by 1, (x_1, x_2, x_3) by (y_1, y_2, y_3), (x_1, x_2) by (y_1, y_2), and so on (order 6).

§1.

Reduction of the Problem.

1. The case of more than two systems of intransitivity may successively be reduced to the case of only two systems. For by combining several systems of intransitivity in one larger system, we can always distribute the letters into two systems, $x_1, x_2 \ldots .$ and $y_1, y_2 \ldots .$, such that the substitutions of I never replace any of the x by any of the y, nor vice versa. We may then deal with these two systems exactly as we did with the systems of intransitivity; the only difference will be that the groups G and H to which we arrive may now themselves be intransitive, but at any rate they possess *a smaller number of systems of intransitivity* than the original intransitive group. We may therefore confine ourselves to the case of two systems of intransitivity, provided we drop the supposition that the constituent groups G and H are transitive.

The example given above exhibits the two extreme types of intransitive groups: in the first group every substitution of G occurs multiplied *by every one* of the substitutions of H, and vice versa; in the last group, each substitution of G occurs multiplied only *by one single* substitution of H, and vice versa. The object of this § is to *reduce the construction of any intransitive group to these two extreme cases.*

2. Let us then suppose we have found an intransitive group I with the given transitive constituents G and H:

$$I = [gh, g'h', g''h'', \ldots .].$$

Let $$l_1 . 1, \ l_2 . 1, \ldots . \ l_\lambda . 1 \qquad (1)$$

be those substitutions of I in which the second factor, h, is unity; they form a group L, since the product* $l.1 \times l'.1 = ll'.1$ belongs again to the substitutions

---

*I may remark that I use, throughout, Jordan's notations; accordingly, in a product the *left-hand* factor is to be applied *first*. Further small characters shall always represent substitutions which belong to the group represented by the corresponding large character.

(1). This group L *is a self-conjugate subgroup*[*] *of* G *as well as of* I; for since every substitution of G is commutative[†] to every substitution of H, the substitution $(gh)^{-1}$ (which is generally equal to $h^{-1}g^{-1}$) may be written $g^{-1}h^{-1}$, and the transformed of $l.1$ by any substitution gh of I, viz. $(gh)^{-1}l.1(gh) = g^{-1}lg.1$, is again one of the substitutions (1).

If gh be any of the substitutions of I, $lg.h$ will also belong to I; and conversely, *if* gh *and* $g'h$ *be two substitutions of* I *with the same factor* h, *then* $g' = lg$; for $g'h \times (gh)^{-1} = g'g^{-1}.1$ belongs to I and in particular to L, therefore $g'g^{-1} = l$ or $g' = lg$.

In like manner the substitutions of I in which the first factor, g, is unity, and which we denote by

$$1.m_1,\ 1.m_2,\ \ldots .\ 1.m_\mu \tag{2}$$

form a group M, self-conjugate in H as well as in I. And if gh and gh' be two substitutions of I with the same factor g, then $h' = mh$.

If l and m be any substitutions of L resp. M, their product $l.m$ will be a substitution of I, and since the substitutions of L are commutative to those of M and both groups have no common substitutions besides unity, the $\lambda.\mu$ products, $l_\alpha m_\beta\,(\alpha = 1,\ 2 \ldots . \lambda;\ \beta = 1,\ 2 \ldots . \mu)$ are all different and form a group J.[‡] *J is a self-conjugate subgroup of I*; for $(gh)^{-1}lm\,(gh) = g^{-1}lg.h^{-1}mh$, and this may be written $l'm'$, since L is self-conjugate in G and M self-conjugate in H.

If in a substitution gh of I the first factor g belongs to L, then the second, h, will belong to M and therefore gh to J. For by multiplying $l.h$ by $l^{-1}.1$, which is sure to belong to I, we see that $1.h$ is a substitution of I and especially of M, therefore $h = m$. Similarly, if the second factor of gh belongs to M, then the first will belong to L.

3. Applying a well-known theorem to the group I and its subgroup J, we exhibit the substitutions of I, whose order we suppose to be r, in the form $l_\alpha m_\beta i_\gamma$, $\alpha = 1,\ 2 \ldots . \lambda;\ \beta = 1,\ 2 \ldots . \mu;\ \gamma = 1,\ 2 \ldots . \varkappa = \dfrac{r}{\lambda\mu}$, the multipliers $i_1 = 1,\ i_2 \ldots . i_\varkappa$ being substitutions of I, subject to the only condition that, if α and β be different, then $i_\beta i_\alpha^{-1}$ shall not belong to the subgroup J.

---

[*] "Ausgezeichnete Untergruppe." See Cole "On Klein's Icosaeder," *Amer. Jour. Math.*, Vol. IX.

[†] "Per mutable à."

[‡] See Serret, Cours d'Algèbre, No. 485.

We decompose the substitutions i_γ into factors g and h and denote them by

$$g_1 h_1 = 1,\ g_2 h_2,\ \ldots .\ g_\varkappa h_\varkappa . \tag{3}$$

The $r = \lambda\mu\varkappa$ substitutions of I may then be written $l_\alpha g_\gamma . m_\beta h_\gamma$, the indices taking the above indicated values.

Since G is supposed to be a transitive constituent of I, the $\lambda\varkappa$ substitutions $l_\alpha g_\gamma$ must all belong to G, and each substitution of G must be found among them. I say these $\lambda\varkappa$ substitutions are all different. For if $l_\alpha g_\gamma$ were equal to $l_{\alpha'} g_{\gamma'}$, we should obtain a relation $g_{\gamma'} = l g_\gamma$; but then the product $i_{\gamma'} i_\gamma^{-1}$ would be equal to $l . h_{\gamma'} h_\gamma^{-1}$ and would, according to No. 2, end, belong to J, against our assumption about the i_γ. Therefore *the $\lambda\varkappa$ substitutions $l_\alpha g_\gamma$ are all different and represent all the substitutions of G and each one only once;* therefore the order p of G is $\lambda\varkappa$.

Similarly the $\mu\varkappa$ substitutions $m_\beta h_\gamma$ are all different and represent exactly the group H, whose order q is consequently $\mu\varkappa$. Hence follows the theorem:

Between the orders p, q, λ, μ of the groups G, H, L, M resp. exists the relation

$$\frac{p}{\lambda} = \frac{q}{\mu} = \varkappa . \tag{4}$$

4. The substitutions $g_1, g_2, \ldots . g_\varkappa$ do not, as a rule, form a group; but we can introduce $\varkappa$ substitutions $\mathfrak{g}_1, \mathfrak{g}_2 \ldots . \mathfrak{g}_\varkappa$ corresponding to them which form a group. In the analytical representation of substitutions of p letters, it is usual to agree that the same letter x_z may also be represented by $x_{z+p}, x_{z+2p} \ldots .$; similarly we will now introduce substitutions of the letters $g_1, g_2 \ldots . g_\varkappa$ and agree that the same letter g_γ may indifferently be represented by g_γ or $l_2 g_\gamma$, $l_3 g_\gamma \ldots .$ or $l_\lambda g_\gamma$. Then the symbol

$$\mathfrak{g}_\gamma = \begin{pmatrix} g_1 & , g_2 & , \ldots . g_\varkappa \\ g_1 g_\gamma, & g_2 g_\gamma, & \ldots . g_\varkappa g_\gamma \end{pmatrix} \tag{5}$$

represents a substitution between the $\varkappa$ letters $g_1, g_2 \ldots . g_\varkappa$, $g_\alpha g_\gamma$ being equivalent to g_β, if $g_\alpha g_\gamma = l g_\beta$. According to a theorem due to Jordan,* the $\varkappa$ substitutions

$$\mathfrak{g}_1, \mathfrak{g}_2, \ldots . \mathfrak{g}_\varkappa$$

---

* Jordan, l. c. No. 69; see besides, König: *Ueber rationale Functionen von n Elementen*, Math. Ann. Vol. 14, and especially Capelli: *Sopra l'isomorfismo dei gruppi di sostituzioni*, Giornale di Matematiche, Vol. 16, pg. 43.

form a regular group* $\mathfrak{G}$, *which is meriedrically isomorphic to* G, the λ substitutions

$$l_1 g_\gamma,\ l_2 g_\gamma,\ \ldots\ l_\lambda g_\gamma$$

of G being coordinated ("zuordnen") with the substitution $\mathfrak{g}_\gamma$ of $\mathfrak{G}$.

In fact the group $\mathfrak{G}$ is identical with the group of those substitutions which undergo, under the action of the substitutions of G, the κ conjugate ("gleichberichtigt") values of a rational function $\phi(x_1 x_2 \ldots)$ which remains unaltered by the substitutions of the subgroup L and no other.

Defining in an analogous way the symbol

$$\mathfrak{h}_\gamma = \begin{pmatrix} h_1\ , & h_2\ , & \ldots & h_\kappa \\ h_1 h_\gamma, & h_2 h_\gamma, & \ldots & h_\kappa h_\gamma \end{pmatrix} \tag{6}$$

as a substitution, we introduce the group

$$\mathfrak{H} = [\mathfrak{h}_1 = 1,\ \mathfrak{h}_2,\ \ldots\ \mathfrak{h}_\kappa],$$

which is regular and meriedrically isomorphic to H, the μ substitutions

$$m_1 h_\gamma,\ m_2 h_\gamma,\ \ldots\ m_\mu h_\gamma$$

of H being coordinated with the substitution $\mathfrak{h}_\gamma$ of $\mathfrak{H}$.

Now we replace in each of the κ multipliers $g_\gamma h_\gamma$ (3), the factor g_γ by the corresponding substitution $\mathfrak{g}_\gamma$, and the factor h_γ by the corresponding substitution $\mathfrak{h}_\gamma$, and obtain in this way a series of κ products,

$$\mathfrak{g}_1 \mathfrak{h}_1 = 1,\ \mathfrak{g}_2 \mathfrak{h}_2,\ \ldots\ \mathfrak{g}_\kappa \mathfrak{h}_\kappa. \tag{7}$$

I say *these κ substitutions* (7) *form a group.* For if $\mathfrak{g}_\alpha \mathfrak{h}_\alpha$ and $\mathfrak{g}_\beta \mathfrak{h}_\beta$ be any two substitutions of the series (7), the series (3) will contain the two products $g_\alpha h_\alpha$ and $g_\beta h_\beta$, and the group I their product $g_\alpha g_\beta . h_\alpha h_\beta$, which we may reduce to the form $lm \times g_\gamma h_\gamma = l g_\gamma . m h_\gamma$, $g_\gamma h_\gamma$ being one of the multipliers (3). Hence we infer

$$g_\alpha g_\beta = l g_\gamma,\quad h_\alpha h_\beta = m h_\gamma.$$

On account of the isomorphic relation between $\mathfrak{G}$ and G, the product $\mathfrak{g}_\alpha \mathfrak{g}_\beta$ in $\mathfrak{G}$ corresponds to the product $g_\alpha g_\beta$ in G; but on the other hand $\mathfrak{g}_\gamma$ corresponds to $l g_\gamma$; therefore since to every substitution of G corresponds but one single substitution of $\mathfrak{G}$, the equation $g_\alpha g_\beta = l g_\gamma$ implies $\mathfrak{g}_\alpha \mathfrak{g}_\beta = \mathfrak{g}_\gamma$, and likewise $\mathfrak{h}_\alpha \mathfrak{h}_\beta = \mathfrak{h}_\gamma$.

But $g_\gamma h_\gamma$ belonging to the series (3), the series (7) contains the product $\mathfrak{g}_\gamma \mathfrak{h}_\gamma = \mathfrak{g}_\alpha \mathfrak{g}_\beta . \mathfrak{h}_\alpha \mathfrak{h}_\beta = \mathfrak{g}_\alpha \mathfrak{h}_\alpha \times \mathfrak{g}_\beta \mathfrak{h}_\beta$, which shows that in fact the substitutions (7) form a group which we will denote by $\mathfrak{J}$.

---

*A substitution-group is said to be *regular*, according to Klein, if it is transitive and its order equals its degree.

This group is evidently *intransitive* and possesses the two systems of intransitivity,

$$g_1, g_2 \dots . g_\kappa \text{ and } h_1, h_2 \dots . h_\kappa,$$

and *its transitive constituents are the groups* $\mathfrak{G}$ and $\mathfrak{H}$.

Moreover $\mathfrak{J}$ is meriedrically isomorphic to I, the $\lambda\mu$ substitutions

$$l_\alpha g_\gamma . m_\beta h_\gamma \; (\alpha = 1, 2 \dots . \lambda; \; \beta = 1, 2 \dots . \mu)$$

of I corresponding to the substitution $\mathfrak{g}_\gamma \mathfrak{h}_\gamma$ of $\mathfrak{J}$; to unity in $\mathfrak{J}$ corresponds the subgroup J of I.

Finally the connection between the group $\mathfrak{J}$ and the multipliers (3) shows plainly that *each substitution of* $\mathfrak{G}$ (*and likewise each substitution of* $\mathfrak{H}$) *occurs only in one single substitution of* $\mathfrak{J}$.

5. *Conversion*: Let us now suppose *we choose at will the self-conjugate subgroups L and M* of the two given groups G and H resp., but such that their orders λ and μ satisfy the relation (4),

$$\frac{p}{\lambda} = \frac{q}{\mu} = \kappa,$$

and form the group J by multiplying their substitutions in all possible ways. Then we derive, as in No. 4, the regular groups $\mathfrak{G}$ and $\mathfrak{H}$ isomorphic to G and H resp. Finally, we suppose it possible to construct an intransitive group

$$\mathfrak{J} = [\mathfrak{g}_1\mathfrak{h}_1, \mathfrak{g}_2\mathfrak{h}_2 \dots . \mathfrak{g}_\kappa\mathfrak{h}_\kappa]$$

with the transitive constituent groups $\mathfrak{G}$ and $\mathfrak{H}$, and such that each substitution of $\mathfrak{G}$ (and each substitution of $\mathfrak{H}$) occurs but in one single substitution of $\mathfrak{J}$.

Repeating, then, in reversed order, the conclusions of the last number, we easily find:

If in each substitution $\mathfrak{g}_\gamma\mathfrak{h}_\gamma$ of $\mathfrak{J}$ we replace $\mathfrak{g}_\gamma$ by any of its corresponding substitutions of G, g_γ, and $\mathfrak{h}_\gamma$ by any of its corresponding substitutions of H, h_γ, and multiply the $\lambda\mu$ substitutions of J by these κ products, $g_1h_1, g_2h_2 \dots . g_\kappa h_\kappa$, the $\lambda\mu\kappa$ substitutions thus obtained form an intransitive group I whose transitive constituents are the given groups G and H.

Thus we have obtained the result announced in the beginning of this §: *The construction of the intransitive group I is indeed resolved into the construction of two intransitive groups which represent the two extreme types* of intransitive groups, viz. *the group J* in which the substitutions of the two constituent groups L and

M appear multiplied together in all possible ways, *and the group* $\mathfrak{J}$ in which each substitution of either of the constituent groups $\mathfrak{G}$ and $\mathfrak{H}$ appears multiplied only by one single substitution of the other constituent group.

§2.

Connection between Intransitive Groups and Isomorphism.

1. Before we pass to the construction of the group $\mathfrak{J}$, we must mention a theorem due to Netto, by which an intimate connection is established between intransitive groups and isomorphism. A group H is said to be *isomorphic, in the most general sense*[*] of the word, to a group G, if it is possible to establish between the two groups a relation of the following character: With each substitution of G are coordinated one or several substitutions of H, and if $h_1, h_2 \ldots\ldots h_\nu$ be all the substitutions *coordinated with a substitution* g, *and* $h'_1, h'_2 \ldots\ldots h'_{\nu'}$ all the substitutions coordinated with any other g', then all the products, $h_\alpha h'_{\alpha'} (\alpha = 1, 2 \ldots\ldots \nu;\ \alpha' = 1, 2 \ldots\ldots \nu')$ and no other substitutions of H, shall be coordinated with the product gg'. Besides, it is supposed that all the substitutions of H are engaged in this coordination.

From this definition *follows* that reciprocally the group G is isomorphic to H, so that the isomorphism between the two groups is perfectly mutual.

The holoedric and the meriedric isomorphism, as defined by Jordan, are special cases of this general isomorphism, which has been called by Netto *mutually meriedric* ("gegenseitig mehrstufig").

Now Netto's theorem[†] is this:

Suppose the two isomorphic groups G and H to operate upon different letters; if, then, each substitution of G is successively multiplied by all its corresponding substitutions of H, the products will form an intransitive group with the constituent groups G and H.

And conversely: If an intransitive group I with the constituent groups G and H be given, and we coordinate with each substitution of G all those substitutions of H with which it appears multiplied in the substitutions of I, then the relation thus established between the two groups will be isomorphic.

2. *Every intransitive group may consequently be considered as defining an isomorphic relation between two groups*, and any theorem on intransitive groups

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[*] Netto: Substitutionentheorie, §93.

[†] Ib. l. c. §93.

may be interpreted as a theorem on isomorphism. For instance, denoting by O_γ the system of λ substitutions, $l_1 g_\gamma, l_2 g_\gamma \ldots . l_\lambda g_\gamma$, and by Ω_γ the system of μ substitutions, $m_1 h_\gamma, m_2 h_\gamma \ldots . m_\mu h_\gamma$, we may express the results of §§1, 3 in this form:

If two groups G and H are isomorphic, the substitutions of G can be distributed into a number of systems

$$O_1, O_2 \ldots . O_\kappa,$$

and those of H in an *equal* number of corresponding systems,

$$\Omega_1, \Omega_2, \ldots . \Omega_\kappa$$

such that any substitution of O_γ is coordinated with all the substitutions of the corresponding system Ω_γ and with no other, and, vice versa, any substitution of Ω_γ with all the substitutions of O_γ.

But this is exactly what Capelli,[*] who has first considered this general kind of isomorphism, *assumes in his definition*, so that by our developments the definitions of Capelli and Netto, though apparently different, are found to be identical.

3. If we apply Netto's theorem to the intransitive group $\mathfrak{J}$ of §1 and remember that in $\mathfrak{J}$ each substitution of $\mathfrak{G}$ appears multiplied only by one substitution of $\mathfrak{H}$ and vice versa, we see that the group $\mathfrak{J}$ defines, between $\mathfrak{G}$ and $\mathfrak{H}$, a holoedric isomorphism, in which the substitution $\mathfrak{g}_\gamma$ of $\mathfrak{G}$ is coordinated with that substitution $\mathfrak{h}_\gamma$ of $\mathfrak{H}$ with which $\mathfrak{g}_\gamma$ appears multiplied in the substitution $\mathfrak{g}_\gamma \mathfrak{h}_\gamma$ of $\mathfrak{J}$.

§3.

Holoedric Isomorphic Relation of a Regular Group to itself.

1. In order to be in accordance with the usual notations, we will write henceforth G and H instead of $\mathfrak{G}$ and $\mathfrak{H}$, and denote the letters upon which the substitutions of G and H operate by $x_1, x_2 \ldots . x_r$ resp. $y_1, y_2 \ldots . y_r$. Since the two groups are supposed to be *regular and holoedrically isomorphic*, they must be, according to a theorem of Jordan's,[†] *altogether identical except*

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[*] Capelli, l. c. pg. 33. Starting from this definition, Capelli proves that the different systems O_γ contain the same number λ of substitutions, and the different systems Ω_γ the same number μ of substitutions, and thus finds the relation (4), $\frac{p}{\lambda} = \frac{q}{\mu}$.

[†] Jordan, l. c. No. 70, 72, and Netto, l. c. §90.

the notation of the letters upon which they operate. Therefore there must exist a certain permutation of the $x: x_{i_1}, x_{i_2} \dots . x_{i_r}$, such that each substitution h_a of H becomes identical with the corresponding substitution g_a of G, if we replace, *within the cycles of* h_a, the letter y_1 by x_{i_1}, y_2 by x_{i_2}, and so on; or denoting by u the substitution

$$u = \begin{pmatrix} x_1, & x_2 \dots . & x_r, & y_1, & y_2 \dots . & y_r \\ y_1, & y_2 \dots . & y_r, & x_{i_1}, & x_{i_2} \dots . & x_{i_r} \end{pmatrix} \tag{8}$$

we must have, between any two corresponding substitutions g_a and h_a, the relation

$$g_a = u^{-1} h_a u, \qquad a = 1, 2 \dots . r \tag{9}$$

and between the groups themselves,

$$G = u^{-1} H u. \tag{10}$$

If there exist still another holoedric isomorphic relation between G and H, there will exist another substitution u' such that

$$G = u'^{-1} H u';$$

u' can be reduced to the form

$$u' = ut,$$

where t is a substitution between the x which transforms G into itself,

$$t^{-1} G t = G. \tag{11}$$

For from $\quad u^{-1} H u = G$ and $u'^{-1} H u' = G$

follows $\quad u'^{-1} u G u^{-1} u' = G,$

or if we put $u^{-1} u' = t$,

$$t^{-1} G t = G$$

and $\quad u' = ut.$

We can write $\quad u = v\tau, \; u' = v'.\tau,$

where

$$\tau = (x_1, y_1)(x_2, y_2) \dots . (x_r, y_r) \tag{12}$$

and v, v' are substitutions of the letters y only; then we see that

$$t = \tau^{-1} (v^{-1} v') \tau$$

is a substitution between the x only.

The equation $t^{-1} G t = G$ defines a holoedric isomorphic relation of G to itself, in which any substitution g is coordinated with its transformed $t^{-1} g t$.

Thus *the construction of the intransitive group I is at last reduced to the problem: To find all possible holoedric isomorphic relations of a regular group to itself.*

2. The substitutions t which satisfy the relation $t^{-1}Gt = G$ form a *group* T, which is *the largest group with the letters* $x_1, x_2 \dots x_r$ *in which the group* G *is contained as a self-conjugate subgroup.*[*]

The group T always contains substitutions which are *commutative* not only to the whole group G, but *to each of its substitutions.* These substitutions form a group which has been studied by Jordan;[†] it is itself regular and of the same order as G and can be derived from it by an algorithm indicated by Jordan. We denote this group by

$$S = [s_1 = 1, s_2, \dots s_r].$$

S being a subgroup of T, we can write the substitutions of T in the form

$$s_\alpha t_\beta, \ \alpha = 1, 2 \dots r, \ \beta = 1, 2 \dots \rho.$$

Two substitutions of T with the same factor t_β will transform the group G exactly in the same way and therefore lead to the same isomorphic relation of G to itself; for

$$(st_\beta)^{-1}g(st_\beta) = t_\beta^{-1}(s^{-1}gs)\,t_\beta,$$

which is equal to $t_\beta^{-1}gt_\beta$, since $s^{-1}gs$ is supposed to be $= g$.

To obtain all the *distinct* isomorphic relations it is therefore sufficient to know a complete system of multipliers, $t_1, t_2 \dots t_\rho$.

The problem to determine the group T for a given regular group G has been completely solved by Jordan[‡] in the case where G is the group of the arithmetical substitutions

$$|z_1, z_2 \dots z_n;\ z_1 + \alpha_1, z_2 + \alpha_2, \dots z_n + \alpha_n|,$$
$$\alpha_1 = 0, 1 \dots m-1,\ \alpha_2 = 0, 1 \dots m-1, \dots \alpha_n = 0, 1 \dots m-1.$$

In this case S is identical with G and the group T is the *linear group of the degree* m^n, which is generated by combining the given group with the group of the geometrical substitutions

$$|z_1, z_2 \dots z_n;\ a_{11}z_1 + a_{12}z_2 + \dots, a_{21}z_1 + a_{22}z_2 + \dots; \dots a_{n_1}z_1 + a_{n_2}z_2 + \dots|,$$

the determinant $|a_{\alpha\beta}|$ being prime to m.

Here the group of the geometrical substitutions form a complete system of multipliers $t_1, t_2 \dots t_\rho$.

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[*] Netto, l. c. §78.

[†] Jordan, l. c. No. 75. See also Frattini: *I gruppi transitivi di sostituzioni dell' istesso ordine e grado.* Atti della R. Acc. dei Lincei, Memorie, 1883.

[‡] Jordan, l. c. No. 19, and Netto, l. c. §137.

3. The isomorphic relation of a group G to itself can be found in a different way without recurring to the group T, provided we know a system of *independent generating substitutions*[*] $a, b, c \dots$ of G and the fundamental relations between them. For if

$$\begin{aligned} a' &= \phi(a, b, c \dots), \\ b' &= \psi(a, b, c \dots), \\ c' &= \chi(a, b, c \dots)\dagger \\ &\dots\dots\dots\dots \end{aligned} \tag{13}$$

be the substitutions which are coordinated with $a, b, c \dots$ resp., then exactly the same relations will exist between the $a', b', c' \dots$ as between the $a, b, c \dots$, and therefore the $a', b', c' \dots$ will constitute another system of generating substitutions for the group G. Conversely, if $a', b', c' \dots$ be another system of independent substitutions satisfying the same fundamental relations (and no other besides), and if we coordinate a' with a, b' with b, c' with c, and so on, and with each product of the $a, b, c \dots$ the similar product of the $a', b', c' \dots$, then we shall have established a holoedric isomorphism of the group G with itself.

For instance, the *cyclic group* of the order n can be generated by a single substitution a, satisfying the relation $a^n = 1$, and no other relation $a^{n'} = 1$ where $n' < n$. The same conditions are satisfied by $a' = a^\mu$, provided μ be prime to n. Therefore we obtain a holoedric isomorphic relation of the cyclic group to itself by coordinating a^α with $a^{\mu\alpha}$, $\alpha = 0, 1, \dots n-1$, and there are $\phi(n)$ different isomorphic relations.

We may even use this method to determine backwards the group T. For by comparing the generating substitutions $a, b, c \dots$ with their corresponding substitutions $a', b', c' \dots$ we can easily find the substitutions t_β of No. 2, and by combining these with the group S we obtain the group T.

Example: The *symmetrical group of three letters* can be generated by two substitutions a, b, satisfying the relations

$$a^3 = 1,\ b^2 = 1,\ ba = a^2 b;$$

the group is then

$$1,\ a,\ a^2,\ b,\ ab,\ a^2 b.$$

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[*] Compare for this number, Cayley, *On the Theory of Groups*, Phil. Mag., 4th series, Vol. 7, 18, and *American Journal of Mathematics*, Vol. 1; and Dyck, *Gruppentheoretische Studien*, Math. Ann., Vol. 22.

† The $\phi, \psi, \chi \dots$ represent products $a^\alpha b^\beta c^\gamma \dots a^{\alpha'} b^{\beta'} c^{\gamma'} \dots$

If we choose for a' any of the substitutions of the third order (a or a^2), for b' any of the substitutions of the second order (b, or ab, or a^2b), it is easily seen that a', b' satisfy the same relations

$$a'^3 = 1\,,\; b'^2 = 1\,,\; b'a' = a'^2b',$$

and no other, and we have therefore six different isomorphic relations of our group to itself. The regular form* of our group is

$$\begin{gathered} 1\,;\; a = (x_1x_2x_3)(x_4x_5x_6)\,;\quad a^2 = (x_1x_3x_2)(x_4x_6x_5)\,;\; b = (x_1x_4)(x_2x_6)(x_3x_5)\,; \\ ab = (x_1x_5)(x_2x_4)(x_3x_6)\,;\; a^2b = (x_1x_6)(x_2x_5)(x_3x_4)\,. \end{gathered}$$

To find the substitution t, for instance, for the isomorphic relation defined by $a' = a$, $b' = ab$, we have to determine a substitution which, applied within the cycles of a and b, changes

$$(x_1x_2x_3)(x_4x_5x_6)$$

into itself, and

$$(x_1x_4)(x_2x_6)(x_3x_5) \text{ into } (x_1x_5)(x_2x_4)(x_3x_6)\,;$$

evidently $(x_1x_2x_3)$ is such a substitution. In this way we find corresponding to the six isomorphic relations, the six substitutions,

$$\begin{gathered} t_1 = 1\,,\; t_2 = (x_1x_2x_3)\,,\; t_3 = (x_1x_3x_2)\,, \\ t_4 = (x_5x_6)(x_2x_3)\,,\; t_5 = (x_5x_6)(x_3x_1)\,,\; t_6 = (x_5x_6)(x_1x_2)\,. \end{gathered}$$

On the other hand the group S is easily found to be

$$\begin{gathered} 1\,;\; (x_1x_2x_3)(x_4x_6x_5)\,,\; (x_1x_3x_2)(x_4x_5x_6)\,,\; (x_1x_4)(x_2x_5)(x_3x_6)\,, \\ (x_1x_5)(x_2x_6)(x_3x_4)\,,\; (x_1x_6)(x_2x_4)(x_3x_5)\,. \end{gathered}$$

These combined with the six t give the group T, which is of the order 36, and is made up of the positive ("gerade") substitutions contained in the most general imprimitive group with the two systems of imprimitivity,

$$x_1,\, x_2,\, x_3 \text{ and } x_4,\, x_5,\, x_6\,.$$

This is then the largest group with the letters $x_1, x_2 \ldots . x_6$ in which G is contained as a self-conjugate subgroup.

4. As an *example of the construction of intransitive groups*, we propose to find all the intransitive groups with the two systems of intransitivity,

$$x_1,\, x_2,\, x_3,\, x_4 \text{ and } y_1,\, y_2,\, y_3,\, y_4,$$

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*That is to say, the regular group which is holoedrically isomorphic to the given group; see Dyck, l. c. §6.

whose transitive constituents are the *alternate group* G_{12} of the letters x_1, x_2, x_3, x_4 and the alternate group H_{12} of the letters y_1, y_2, y_3, y_4.

The group G_{12} can be generated by two substitutions a and c, satisfying the relations

$$a^2 = 1, \; c^3 = 1, \; (ac)^3 = 1. \tag{14}$$

Introducing, for shortness, the notation

$$b = cac^2,$$

the group G_{12} may be exhibited in the following table:

$$\begin{array}{llll} 1, & a\,, & b\,, & ab\,, \\ c\,, & ac\,, & bc\,, & abc\,, \\ c^2, & ac^2, & bc^2, & abc^2, \end{array} \tag{15}$$

whose first line represents the "Four-group" ("Vierergruppe"), which shall be denoted by G_4, and which is, besides G_{12} itself and $G_1 = 1$, the only self-conjugate subgroup of G_{12}.

We may choose, for instance,

$$a = (x_1x_2)(x_3x_4), \; c = (x_1x_2x_3),$$

and we obtain all the possible systems a', c' of generating substitutions which satisfy the same fundamental relations, by choosing for a' any of the three substitutions of the second order, a, b, ab, and for c' any of the eight substitutions of the third order,

$$c, ac, bc, abc; \; c^2, ac^2, bc^2, abc^2.$$

For since c' does not belong to the subgroup G_4, whereas a' does, $a'c'$ cannot belong to G_4 and must therefore be one of the eight substitutions of the third order, therefore we have

$$a'^2 = 1, \; c'^3 = 1, \; (a'c')^3 = 1. \tag{16}$$

And there can exist *no other relation* between a' and c' independent of these. For any such relation could, by means of (16), be reduced to the form: one of the products of the table (15), marked with dashes, equal to unity (b' denoting $c'a'c'^2$). Let us for instance suppose we had besides (16) the relation $a'b'c' = 1$; now $a'b'c'$ is conjugate ("gleichberechtigt") in G_{12} with c', since, on account of (16), $a'b'c' = a'^{-1}c'a'$; therefore the relation $a'b'c' = 1$ implies* $c' = 1$, which is in contradiction with our assumption about c'. Since the substitutions a, b, ab are conjugate in G_{12}, and likewise the substitutions of the second line of (15), and finally those of the third line, we see that any relation between a' and c', independent of (16), would imply either $a' = 1$, or $c' = 1$ or $c'^2 = 1$.

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* Compare for these conclusions Dyck, l. c. §3.

There exist therefore $3.8 = 24$ different systems of generating substitutions satisfying the relations (14) and no other.

Denoting the generating substitutions of H_{12} by d, f, we have

$$d^2 = 1,\ f^3 = 1,\ (df)^3 = 1.$$

And putting again $e = fdf^2$, the group H_{12} may be exhibited in the table

$$\begin{array}{cccc} 1\ , & d\ , & e\ , & de\ , \\ f\ , & df, & ef, & def\ , \\ f^2, & df^2, & ef^2, & def^2, \end{array} \tag{17}$$

whose first line shall be denoted by H_4.

On account of the relation (4) we can choose the self-conjugate subgroups L, M of §1 in three different ways: either G_{12}, H_{12} $(x = 1)$, or G_4, H_4 $(x = 3)$, or G_1, H_1 $(x = 12)$.

a). $x = 1$; $L = G_{12}$, $M = H_{12}$.

I is of the order 144, and is obtained by multiplying every substitution of G_{12} by every substitution of H_{12}.

b). $x = 3$; $L = G_4$, $M = H_4$.

The multipliers g_1, g_2, g_3 of §1, 3 are here:

$$g_1 = 1,\ g_2 = c,\ g_3 = c^2,$$

hence

$$\mathfrak{g}_1 = 1,$$

$$\mathfrak{g}_2 = \begin{pmatrix} g_1 & , g_2 & , g_3 \\ g_1g_2, & g_2g_2, & g_3g_2 \end{pmatrix} = (g_1,\ g_2,\ g_3) = \mathfrak{g},$$

$$\mathfrak{g}_3 = \begin{pmatrix} g_1 & , g_2 & , g_3 \\ g_1g_3, & g_2g_3, & g_3g_3 \end{pmatrix} = (g_1,\ g_3,\ g_2) = \mathfrak{g}^2,$$

$$\mathfrak{G} = [1,\ \mathfrak{g},\ \mathfrak{g}^2].$$

In the isomorphism between $\mathfrak{G}$ and G_{12}, 1 is coordinated with the first line of the table (15), $\mathfrak{g}$ with the second, $\mathfrak{g}^2$ with the third.

Further,

$$h_1 = 1,\ h_2 = f,\ h_3 = f^2,$$

$$\mathfrak{h}_1 = 1,\ \mathfrak{h}_2 = (h_1,\ h_2,\ h_3) = \mathfrak{h};\ \mathfrak{h}_3 = (h_1,\ h_3,\ h_2) = \mathfrak{h}^2,$$

$$\mathfrak{H} = [1,\ \mathfrak{h},\ \mathfrak{h}^2].$$

The substitution 1 of $\mathfrak{H}$ is coordinated with the first line of the table (17), $\mathfrak{h}$ with the second, $\mathfrak{h}^2$ the third.

There exist two intransitive groups $\mathfrak{J}$ of the third order with the transitive constituents $\mathfrak{G}$ and $\mathfrak{H}$ (§3, 3), viz.

$$1,\ \mathfrak{g}\mathfrak{h},\ \mathfrak{g}^2\mathfrak{h}^2,$$

and

$$1,\ \mathfrak{g}\mathfrak{h}^2,\ \mathfrak{g}^2\mathfrak{h}.$$

Consequently we obtain *two intransitive groups I of the order* 48: the first by multiplying every substitution of the first line of (15) by every substitution of the first line of (17), every one of the second by every one of the second, every one of the third by every one of the third; the second group I by multiplying every substitution of the first line of (15) by every substitution of the first line of (17), every one of the second by every one of the third, every one of the third by every one of the second.

c). $\varkappa = 12;\ L = 1,\ M = 1.$

Here $\mathfrak{G}$ is nothing but the regular form of the group G_{12}, $\mathfrak{H}$ the regular form of the group H_{12}. The 24 systems of generating substitutions of G_{12} define as many holoedric isomorphic relations of G_{12} (and therefore also of $\mathfrak{G}$) to itself, and consequently we obtain 24 *groups I of the order* 12, all contained in the table:

$$\begin{array}{llll}
1 & ,\ a'.d & ,\ b'.e & ,\ a'b'.de, \\
c'.f & ,\ a'c'.df & ,\ b'c'.ef & ,\ a'b'c'.def, \\
c'^2.f^2, & a'c'^2.df^2, & b'c'^2.ef^2, & a'b'c'^2.def^2,
\end{array}$$

if we replace a', c' successively by the 24 different systems of generating substitutions.

§4.

Some Applications to the Construction of Transitive Groups.

1. In a note published in the Mathematische Annalen,* Cayley proves a theorem which is closely connected with our subject, and which, in our notations, may be expressed thus:

"Let $G = [g_1, g_2 \ldots g_r]$ be a substitution-group with the letters $x_1, x_2 \ldots$, whose substitutions are *commutative each to each,* and let h_α be the same substitution as g_α but operating upon different letters $y_1, y_2 \ldots$, and let τ denote the substitution

$$\tau = (x_1y_1)(x_2y_2)(x_3y_3)\ \ldots\ ,$$

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* Cayley, *A Theorem on Groups*, Math. Ann., Vol. 13.

then the $2r$ substitutions

$$\begin{array}{l} g_1 h_1^m \;\;, g_2 h_2^m \;, \ldots\ldots g_r h_r^m, \\ g_1 h_1^m . \tau, \; g_2 h_2^m \tau, \ldots\ldots g_r h_r^m . \tau \end{array}$$

will always form a group, if

$$m^2 \equiv 1 \qquad (\text{mod. } r)."$$

The first line of this group is an intransitive group of the special class considered in the last §, and we propose therefore to apply the developments of §3 to the problem:

Suppose the two regular groups G and H of the order r of §3, 1 to be identical, but operating upon different letters x resp. y; h_α shall denote the same substitution of $y_1, y_2 \ldots\ldots y_r$ which g_α denotes of the $x_1, x_2, \ldots\ldots x_r$. Let then

$$I = [g_1 h_{i_1}, g_2 h_{i_2} \ldots\ldots g_r h_{i_r}]$$

be an intransitive group of the same order r with the transitive constituents G and H; $i_1, i_2, \ldots\ldots i_r$ is then a certain permutation of $1, 2 \ldots\ldots r$ and there exists, according to §3, 1, a substitution u such that

$$u^{-1} h_{i_\alpha} u = g_\alpha, \quad \alpha = 1, 2 \ldots\ldots r. \tag{18}$$

Now we multiply the substitutions of I by the substitution

$$\tau = (x_1 y_1)(x_2 y_2) \ldots\ldots (x_r y_r),$$

it is required to find the conditions that the $2r$ substitutions

$$\begin{array}{l} g_1 h_{i_1} \;\;, \; g_2 h_{i_2} \quad \ldots\ldots g_r h_{i_r}, \\ g_1 h_{i_1} . \tau, \; g_2 h_{i_2} . \tau \ldots\ldots g_r h_{i} \,. \tau \end{array} \tag{19}$$

form a group.

If they do form a group Q, this group will be imprimitive with the two systems of imprimitivity $x_1, x_2 \ldots\ldots x_r$ and $y_1, y_2 \ldots\ldots y_r$, and I will consist of all those substitutions of Q which do not interchange the systems, and therefore I will be a self-conjugate[*] subgroup of Q, and therefore

$$\tau^{-1} I \tau = I, \tag{20}$$

and this condition is sufficient, since then the two groups I and $[1, \tau]$, which contain no common substitution besides 1, are interchangeable among each other.[†]

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[*] See Netto, l. c. §72.

[†] See Serret, Cours d'Algèbre, No. 435.

From (20) follows that $\tau^{-1}I\tau$ and I must possess the same constituent groups, therefore $\tau^{-1}H\tau = G$; τ and u being two substitutions which transform H into G, they must, according to §3, 1, be connected by an equation

$$u = \tau t, \tag{21}$$

t being commutative to G. Remembering, then, that according to the adopted notation

$$\tau^{-1}h_{t_\alpha}\tau = g_{t_\alpha},$$

(18) becomes

$$t^{-1}g_{t_\alpha}t = g_\alpha, \quad \alpha = 1, 2 \ldots . r. \tag{22}$$

On the other hand, $\tau^{-1}g_\alpha h_{i_\alpha}\tau$, which is equal to $\tau^{-1}g_\alpha\tau . \tau^{-1}h_{t_\alpha}\tau = g_{t_\alpha}h_\alpha$ belongs again to I (20), and therefore g_{i_α} and h_α must satisfy (18),

$$u^{-1}h_\alpha u = g_{i_\alpha},$$

or

$$t^{-1}g_\alpha t = g_{i_\alpha}, \quad \alpha = 1, 2 \ldots . r. \tag{23}$$

This equation combined with (22) gives

$$t^{-2}g_\alpha t^2 = g_\alpha, \quad \alpha = 1, 2 \ldots . r. \tag{24}$$

If, conversely, t satisfy this condition and we choose $h_{i_\alpha} = \tau t g_\alpha t^{-1}\tau^{-1}$, then we find $\tau^{-1}I\tau = I$. We see therefore that the square of the substitution t must belong to the group denoted in §3, 2 by S. For instance, in the case of the group of the arithmetical substitutions mentioned in §3, 2, we find easily for the substitution (see pg. 23):

$$t = | z_1, z_2 \ldots z_n ;\ a_{11}z_1 + a_{12}z_2 + \ldots ,\ a_{21}z_1 + a_{22}z_2 + \ldots ,\ \ldots\ a_{n1}z_1 + a_{n2}z_2 + \ldots |$$

the condition

$$\sum_{\beta=1}^{n} a_{\alpha\beta}a_{\beta\gamma} \equiv \begin{cases} 0, \text{ if } \gamma \gtrless \alpha \\ 1, \text{ if } \gamma = \alpha \end{cases} \pmod{m}.$$

The relation (24) shows that the isomorphic relation of G to itself as defined by the equation

$$t^{-1}Gt = G,$$

possesses the *character of an involution,* that is to say, if g_β in the transformed group is coordinated with g_α in the original group, then reciprocally g_β in the original group will be coordinated with g_α in the transformed group.

Applying this to the generating substitutions $a, b, c \dots$ of §3. 3, we see that the transformation (13) must possess the period 2, that is,

$$\begin{aligned} a &= \phi(a', b', c' \dots), \\ b &= \psi(a', b', c' \dots), \\ c &= \chi(a', b', c' \dots). \\ &\dots\dots\dots\dots \end{aligned}$$

Let us now return to Cayley's theorem and suppose that any two substitutions of G are commutative to each other. Then, according to a theorem of Kronecker's,[*] G can be generated by a series of substitutions $a, b, c \dots$, commutative each to each and of the orders r_1, r_2, r_3 resp., the integers $r_1, r_2, r_3 \dots$ forming a series in which *each integer is divisible by the following*, and besides their product equals the order of G:

$$r = r_1 . r_2 . r_3 \dots .$$

By coordinating each substitution g_α with g_α^m, we always obtain a holoedric isomorphic relation of G to itself, if m be prime to r, since $(g_\alpha g_\beta)^m = g_\alpha^m g_\beta^m$ in this case. The corresponding transformation (13) of the generating substitutions is

$$a' = a^m, \quad b' = b^m, \quad c' = c^m, \dots ,$$

and this has the period 2, then and only then, when

$$a^{m^2} = a, \quad b^{m^2} = b, \quad c^{m^2} = c, \dots ,$$

which implies the only condition,

$$m^2 \equiv 1 \ (\text{mod. } r_1),$$

since $r_2, r_3, \dots$ are divisors of r.

This congruence includes Cayley's theorem, since r is divisible by r_1.

2. We conclude with some general remarks on the construction of groups.

The fundamental problem of the abstract theory of groups is, *to construct all the groups of a given order;* groups which are holoedrically isomorphic may be considered as essentially identical in this research.

Now all possible groups of a given order r may be divided into *two classes:*

The first class contains all those groups which are *holoedrically isomorphic with intransitive groups whose transitive constituents are all of a smaller order than r.*

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[*] Kronecker, Berliner Monatsberichte, 1870, and Netto, l. c. §133.

The second class contains all those groups for which no such isomorphism exists.

All the groups of the first class can be obtained by the method developed in this paper, provided all the groups of a smaller order than r have been previously determined, and thus the problem is reduced to the construction of the groups of the second class.

Examples:

1). $r = 4$. There are two groups,

α).
$$1, a, a^2, a^3;\ a^4 = 1,$$

which belongs to the second class, and

β).
$$1, a, b, ab;\ a^2 = 1,\ b^2 = 1,\ ba = ab,$$

which belongs to the first class, since we may choose

$$a = (x_1x_2),\ b = (y_1y_2).$$

2). $r = 6$. There are two groups,

α).
$$1, a, a^2, a^3, a^4, a^5;\ a^6 = 1$$

belongs to the first class and becomes identical with the intransitive group

$$b = (x_1x_2),\ c = (y_1y_2y_3),$$
$$1, c, c^2, b, bc, bc^2,$$

if we put $a = bc$, which is indeed of the sixth order.

β). The group
$$1, a, a^2, b, ab, a^2b,$$
$$a^3 = 1,\ b^2 = 1,\ ba = a^2b,$$

belongs to the second class, but can be obtained by Cayley's theorem. For if we put

$$c = (x_1x_2x_3),\ d = (y_1y_2y_3),\ \tau = (x_1y_1)(x_2y_2)(x_3y_3),$$

the group
$$1,\ cd^2,\ c^2d,$$
$$\tau,\ cd^2\tau,\ c^2d\tau$$

becomes identical with the given group if we write

$$a = cd^2,\ b = \tau.$$

3). $r = 8$. There exist five different groups enumerated by Cayley in the Phil. Mag. (4th series, Vol. 7, 18). Two of them belong to the first class.

In the group

$$a^4 = 1\,,\ b^2 = 1\,,\ ba = ab$$

we may choose $a = (x_1 x_2 x_3 x_4)\,,\ b = (y_1 y_2)\,,$

and in the group $a^2 = 1\,,\ b^2 = 1\,,\ c^2 = 1\,,$

$$ab = ba\,,\ ac = ca\,,\ bc = cb$$

we may choose $a = (x_1 x_2)\,,\ b = (y_1 y_2)\,,\ c = (z_1 z_2)\,.$

Finally, the group $a^4 = 1\,,\ b^2 = 1\,,\ ab = ba^3$

belongs to the second class, but can be obtained by Cayley's theorem. For if we put

$$c = (x_1 x_2 x_3 x_4)\,,\ d = (y_1 y_2 y_3 y_4)\,,\ \tau = (x_1 y_1)(x_2 y_2)(x_3 y_3)(x_4 y_4)\,,$$

the group

$$\begin{array}{l} 1\,,\ cd^3\,,\ c^2d^2\,,\ c^3d\,, \\ \tau\,,\ cd^3\tau\,,\ c^2d^2\tau\,,\ c^3d\tau \end{array}$$

becomes identical with the given group if we write

$$a = cd^3,\ b = \tau.^*$$

ORANGE, N. J., September, 1888.

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[*] I avail myself of this opportunity to add a reference to a previous paper "On binary sextics with linear transformations into themselves" (this Journal, Vol. X, pg. 47): Mr. Wiltheiss has called my attention to a paper of his (Math. Ann., Vol. 21, pg. 398) which had escaped my notice, and in which he deals with the determination of those systems of hyperelliptic ϑ-moduli for which there exists a complex multiplication. It is easy to prove *a priori* that for the case of the transformation of the first degree (for which Mr. Wiltheiss gives a complete table) these systems of ϑ-moduli must be identical with those determined in the third section of my paper, and so I could have made use of Mr. Wiltheiss's results and determined afterwards the order in which these systems of ϑ-moduli are to be coordinated with the binary sextics, by considering the isomorphism which exists between the group of the linear transformation of the variables z_1, z_2 and the group of the linear transformation of the ϑ-moduli. But still the method employed in my paper has the advantage of showing directly the connection between the binary sextic and the corresponding ϑ-moduli. This method has, by the by, been first employed by Mr. Poincaré in a similar research.

Die Herstellung einer linearen Differentialgleichung aus einem gegebenen Element der Integralfunction.

Von Karl Heun, *München.*

Es kommt nicht selten vor, dass ein Element einer analytischen Function einer unabhängigen Veränderlichen in Form einer convergenten Potenzreihe gegeben ist, deren Coefficienten durch eine homogene lineäre Recursionsgleichung bestimmt sind. In diesem Falle kann man das Problem der Integration regulärer lineärer Differentialgleichungen durch Potenzreihen umkehren und die Differentialgleichung bestimmen, welcher die gegebene Reihe genügt. Diese elementare Aufgabe ist im Folgenden gelöst. Bemerkenswerth scheint mir der äusserst Kurze und durchsichtige Beweis für die Existenz der Integrale regulärer Differentialgleichungen, welcher als ein einfaches Corollar der nachstehenden Lösung erscheint. Dass der hier gegebene Beweis ohne Benutzung des Integralalgorithmus geführt ist, erscheint mir ebenfalls als ein Vorzug desselben.

1.

Ein Element einer analytischen Function y des Argumentes x sei gegeben durch die in der Umgebung des Punktes a convergente Reihe

$$y = g_0 + g_1(x-a) + g_2(x-a)^2 + \dots + \text{in inf.}$$

Ferner seien die Coefficienten $g_0, g_1, g_2, \dots$ durch die recurrente Relation

$$G_0(n).g_n + G_1(n-1).g_{n-1} + G_2(n-2).g_{n-2} + \dots + G_{n-r}(n-r).g_{n-r} = 0 \quad (1)$$

verknüpft. Wir denken uns die Grössen $G_0, G_1, \dots G_{n-r}$ als ganze rationale Functionen p^{ten} Grades von n gegeben. Einige derselben können jedoch auch von einem niederen Grade sein. Erreicht G_0 wirklich den Grad p, dann bringe man die Gleich. (1) auf die Form

$$\frac{g_n}{g_{n-1}} + \frac{G_1}{G_0} + \frac{G_2}{G_0} \cdot \frac{g_{n-2}}{g_{n-1}} + \dots + \frac{G_r}{G_0} \cdot \frac{g_{n-r}}{g_{n-1}} = 0 \qquad \text{(1a)}$$

und setze

$$\frac{G_r}{G_0} = \sum_{\nu=0}^{\nu=\infty} \gamma_\nu^{(r)} \left(\frac{1}{n}\right)^\nu. \tag{a}$$

Wir versuchen nun die Gleichung (1a) durch die Annahme

$$\frac{g_n}{g_{n-1}} = \sum_{\nu=0} c_\nu \left(\frac{1}{n}\right)^\nu = \mathfrak{P}\left(\frac{1}{n}\right)$$

identisch zu befriedigen. Dann ist

$$\frac{g_{n-1}}{g_{n-2}} = \mathfrak{P}\left(\frac{1}{n-1}\right),$$

$$\frac{g_{n-1}}{g_{n-3}} = \mathfrak{P}\left(\frac{1}{n-1}\right) . \mathfrak{P}\left(\frac{1}{n-2}\right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Die Gleich. (1a) geht durch Einführung dieser Potenzreihen über in

$$\mathfrak{P}\left(\frac{1}{n}\right) . \mathfrak{P}\left(\frac{1}{n-1}\right) \cdots\cdot \mathfrak{P}\left(\frac{1}{n-r+1}\right)$$
$$+ \frac{G_1}{G_0} . \mathfrak{P}\left(\frac{1}{n-1}\right) \mathfrak{P}\left(\frac{1}{n-2}\right) \cdots\cdot \mathfrak{P}\left(\frac{1}{n-r+1}\right) + \cdots\cdot + \frac{G_r}{G_0} = 0. \tag{1b}$$

Hieraus lassen sich die Grössen C_0, C_1, ... nach der Methode der unbestimmten Coefficienten berechnen. Für C_0 erhält man die Gleichung

$$C_0^r + \gamma_0^{(1)} . C_0^{r-1} + \gamma_0^{(2)} . C_0^{r-2} + \cdots\cdot + \gamma_0^{(r-1)} . C_0 + \gamma_0^{(r)} = 0. \tag{2}$$

Die Wurzeln derselben seien $\alpha_1, \alpha_2, \ldots . \alpha_r$. Man setze nun für $\mathfrak{r} = 1, 2, \ldots . r$:

$$\alpha_\mathfrak{r} = \frac{1}{\xi_\mathfrak{r} - a} . \tag{3}$$

Dann ist also

$$\frac{g_n}{g_{n-1}} = \frac{1}{\xi_\mathfrak{r} - a} + \frac{C_1}{n} + \frac{C_2}{n^2} + \cdots .$$

Nach einem bekannten von Herrn Weierstrass auf complexe Grössen übertragenen Gauss'schen Satze* ist also die Reihe

$$y = g_0 + g_1(x-a) + g_2(x-a)^2 + \cdots . + \text{ in inf.}$$

absolut convergent für alle Werthe von x, welche der Bedingung genügen

$$|x-a| < |\xi_\mathfrak{r} - a| .$$

---

* Gauss' Werke, Vol. III, pag. 139-143. Weierstrass: Abhandlungen aus der Functionenlehre, p. 212-225.

Jetzt ist natürlich unter $\xi_{\mathfrak{r}}$ derjenige der durch Gleich. (2) bestimmten Punkte $\xi_1, \xi_2, \dots . \xi_r$ zu verstehen, welcher dem Punkte a am *nächsten* liegt. Werden mehrere Wurzeln der Gleichung (2) einander gleich, dann ändert sich nichts an der vorstehenden Schlussfolgerung. Wird $a_{\mathfrak{r}} = 0$ $[r = 1]$ dann ist $\xi_{\mathfrak{r}} = \infty$ zu setzen.

2.

Wir wollen jetzt die lineäre Differentialgleichung aufstellen, welcher die Reihe y formal genügt. Zu diesem Zwecke setze* man

$$\begin{aligned}G_{\mathfrak{r}}(n-\mathfrak{r}) = G_{\mathfrak{r}}(0) + \Delta G_{\mathfrak{r}}(0)(n-\mathfrak{r}) + \frac{1}{1.2}\Delta^2 G_{\mathfrak{r}}(0)(n-\mathfrak{r})(n-\mathfrak{r}-1)\\ + \dots . + \frac{1}{1.2\dots . p}\Delta^p G_{\mathfrak{r}}(0)(n-\mathfrak{r}) \dots . (n-\mathfrak{r}-p+1).\end{aligned}$$

Dann lässt sich die Gleich. (1) in die Form bringen:

$$\begin{aligned}&G_0(0).g_n + G_1(0).g_{n-1} + \dots . + G_r(0).g_{n-r}\\ &+ \Delta G_0(0).n.g_n + \Delta G_1(0).n-1.g_{n-1} + \dots . + \Delta G_r(0).n-r.g_{n-r}\\ &+ \frac{1}{1.2}[\Delta^2 G_0(0).n.n-1.g_n + \Delta^2 G_1(0).n-1.n-2.g_{n-1}\\ &+ \dots . + \Delta G_r(0).n-r.n-r-1.g_{n-r}]\\ &+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ &+ \frac{1}{1.2\dots . p}[\Delta^p G_0(0).n.n-1\dots . n-p+1.g_n\\ &+ \dots . + \Delta^p G_r(0).n-r.n-r-1\dots . n-r-p+1.g_{n-r}] = 0.\end{aligned}$$

Diese Gleichung multiplicire man mit $(x-a)^n$ und summire von $n = 0$ bis $n = \infty$. Beachtet man ferner, dass g mit einem negativen Index verschwinden muss, *falls* die Gleich. (1) auch die Coefficienten $g_1, g_2, \dots . g_{r-1}$ bestimmen soll, dann erhält man den Ausdruck

$$\begin{aligned}&[G_0(0) + G_1(0)(x-a) + G_2(0)(x-a)^2 + \dots . + G_r(0)(x-a)^r].y\\ &+ [\Delta G_0(0) + \Delta G_1(0)(x-a) + \Delta G_2(0)(x-a)^2 + \dots . + \Delta G_r(0)(x-a)^r](x-a).\frac{dy}{dx}\\ &+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ &+ \frac{1}{1.2\dots . p}[\Delta^p G_0(0) + \Delta^p G_1(0)(x-a) + \Delta^p G_2(0)(x-a)^2\\ &\qquad + \dots . + \Delta^p G_r(0)(x-a)^r](x-a)^p.\frac{d^p y}{dx^p} = 0. \qquad \text{(I)}\end{aligned}$$

---

* Die Operation Δ ist in der üblichen Weise zu verstehen. Es ist also

$$\Delta f(x) = f(x+1) - f(x), \quad \Delta^2 f(x) = \Delta f(x+1) - \Delta f(x), \text{ etc.}$$

Die Reihe y genügt also einer inbezug auf den Punkt a *regulären* linearen Differentialgleichung p^{ter} Ordnung. An der Stelle $x = a$ findet im Allgemeinen eine Verzweigung statt. Da nun infolge der Voraussetzung über den Geltungsbereich der Relation (1) die Gleichung: $G_0(0).g_0 = 0$ bestehen muss, welchen Werth man auch g_0 beilege, so ist $G_0(0)$ gleich Null anzunehmen. Die determinirende Gleichung inbezug auf den Punkt $x = a$ heisst dann

$$0 = \Delta G_0(0) + \frac{1}{1.2}\Delta^2 G_2(0)(\lambda - 1) + \frac{1}{1.2.3}\Delta^3 G_0(0)(\lambda - 1)(\lambda - 2)$$
$$+ \ldots\ldots + \frac{1}{1.2\ldots\ldots p}\Delta^p G_0(0)(\lambda - 1)(\lambda - 2)\ldots\ldots(\lambda - p + 1). \quad \text{(Ia)}$$

Die übrigen Verzweigungspunkte der Integralfunction von Gleich. (I) sind Wurzeln der Gleichung

$$\Delta^p G_0(0) + \Delta^p G_1(0)(x - a) + \Delta^p G_2(0)(x - a)^2 + \ldots\ldots + \Delta^p G_r(0)(x - a)^r = 0. \quad (3)$$

Umgekehrt ist bekanntlich nicht jeder Werth von x, welcher diese Gleichung identisch befriedigt, nothwendig einem Verzweigungspunkte entsprechend. Es ist jedoch nicht nöthig auf diese Verhältnisse hier näher einzugehen, da dieselben aus der allgemeinen Theorie der Differentialgleichungen hinreichend bekannt sind.

Als Beispiel der Construction der Gleich. (I) aus einer gegebenen Reihe möge das folgende dienen. Aus der Gauss'schen Reihe $F(\alpha, \beta, \gamma, x)$ leite man eine erhebt andere ab indem man jeden Coefficienten von $x^0, x^1, x^2, \ldots\ldots$ zum Quadrat erhebt. So entsteht die Reihe

$$\Phi(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha^2.\beta^2}{\gamma^2}x + \frac{\alpha^2(\alpha+1)^2.\beta^2(\beta+1)^2}{1^2.2^2.\gamma^2(\gamma+1)^2}x^2 + \ldots\ldots + \text{in inf.}$$

Man soll die Differentialgleichung aufstellen, welche die Function $\Phi(\alpha, \beta, \gamma, x)$ identisch befriedigt. Die Relation (1) heisst in diesem Falle

$$n^2(\gamma + n - 1)^2 g_n - (\alpha + n - 1)^2(\beta + n - 1)^2 g_{n-1} = 0.$$

Es ist also

$$G_0(n) \qquad = (\gamma - 1)^2 n^2 + 2(\gamma - 1)n^3 + n^4,$$
$$-G_1(n - 1) = [\alpha\beta + (\alpha + \beta)(n - 1) + (n - 1)^2]^2.$$

Folglich wird

$$
\begin{aligned}
\Delta G_0(n) &= \gamma^2 + 2\gamma(\gamma+1)n + 6\gamma n^2 + 4n^3,\\
-\Delta G_1(n-1) &= (\alpha+\beta+1)(2\alpha\beta+\alpha+\beta+1) + 2[(\alpha+\beta)^2+3(\alpha+\beta)+2\alpha\beta+2](n-1)\\
&\qquad + 6(\alpha+\beta+1)(n-1)^2 + 4(n-1)^3,\\
\Delta^2 G_0(n) &= 2\gamma(\gamma+4) + 4 + 12(\gamma+1)n + 12n^2,\\
\Delta^3 G_0(n) &= 12(\gamma+2) + 24n,\\
\Delta^4 G_0(n) &= 24,\\
-\Delta^2 G_1(n-1) &= 2[(\alpha+\beta)^2+6(\alpha+\beta)+7] + 12(\alpha+\beta+2)(n-1) + 12(n-1)^2,\\
-\Delta^3 G_1(n-1) &= 12(\alpha+\beta+3) + 24(n-1),\\
-\Delta^4 G_1(n-1) &= 24.
\end{aligned}
$$

Die Differentialgleich. (I) erhält also die Form

$$
\begin{aligned}
(1-x)x^3 . \frac{d^4y}{dx^4} &+ 2[\gamma + 2 - (\alpha+\beta+3)x]\, x^2 . \frac{d^3y}{dx^3}\\
&+ \{\gamma(\gamma+4) + 2 - [(\alpha+\beta)^2 + 6(\alpha+\beta) + 7]\, x\}\, x . \frac{d^2y}{dx^2}\\
&+ [\gamma^2 - (\alpha+\beta+1)(2\alpha\beta+\alpha+\beta+1)\, x] . \frac{dy}{dx} - \alpha^2\beta^2 . y = 0. \qquad \text{(I')}
\end{aligned}
$$

Für das Schema der Verzweigungsindices findet man:

$$
\left\{
\begin{array}{ccc}
\underline{0} & \underline{1} & \underline{\infty}\\
0\,, & 0\,, & \alpha\\
0\,, & 1\,, & \alpha\\
1-\gamma, & 2\,, & \beta\\
1-\gamma, & 2(\gamma-\alpha-\beta)+1, & \beta
\end{array}
\right\}
$$

Die Punkte $x = 0$ und $x = 1$ sind also in der That Verzweigungspunkte für die Integralfunction von Gleich. (I'). Das vollständige Integralsystem der Differentialgleichung (I') hat in der Umgebung des Punktes $x = 0$ die Form

$$
\left.
\begin{aligned}
y_{11} &= \Phi(\alpha, \beta, \gamma, x),\\
y_{21} &= y_{11} . \lg x + \mathfrak{P}_1(x),\\
y_{31} &= x^{1-\gamma} . \Phi(\alpha-\gamma+1,\ \beta-\gamma+1,\ 2-\gamma,\ x),\\
y_{41} &= y_{31} . \lg x + \mathfrak{P}_2(x),
\end{aligned}
\right\}\ \text{für } (x) < 1.
$$

Weniger einfach sind die zu den Punkten $x = 1$ und $x = \infty$ gehörigen Integrale.

3.

In der Gleich. (2), welche die Grösse C_0 bestimmt setze man für

$$\gamma_0^{(1)}, \gamma_0^{(2)}, \ldots . \gamma_0^{(r)}$$

ihre Werthe ein, nemlich

$$\gamma_0^{(\mathfrak{r})} = \frac{\Delta^p G_{\mathfrak{r}}(0)}{\Delta^p G_0(0)}$$

dann geht dieselbe über in

$$0 = \Delta^p G_r(0) + \Delta^p G_{r-1}(0) . C_0 + \Delta^p G_{r-2}(0) . C_0^2 + \ldots . \\ + \Delta^p G_1(0)\, C^{r-1} + \Delta^p G_0(0) . C_0^r . \quad \text{(2a)}$$

Die Grössen $\xi_1, \xi_2, \ldots . \xi_{\mathfrak{r}}, \ldots . \xi_r$ in Gleich. (3) sind also geradezu die Wurzeln der Gleich. (3). Die Reihe

$$y = g_0 + g_1(x-a) + g_2(x-a)^2 + \ldots . + \text{ in inf.}$$

convergirt also unbedingt innerhalb eines Kreises, dessen Mittelpunkt der Punkt $x = a$ ist und dessen Peripherie durch den nächsten Nullpunkt der Function

$$\Delta^p G_0(0) + \Delta^p G_1(0)(x-a) + \ldots . + \Delta^p G_r(x-a)^r$$

hindurchgeht.

Hiermit ist der Existenz bereich der Integrale homogener regulärer linearer Differentialgleichungen fest gestellt (cf. die bekannte Abh. des Herrn Fuchs in Journ. f. Math., t. 66, und die Arbeit des Herrn Frobenius, Ueber die Integration der linearen Differentialgleichungen durch Reihen, ib. t. 76).

Will man nach den vorstehenden Principien das Verhalten der Reihenentwicklungen auf den Convergenzkreisen und insbesondere in den Verzweigungspunkten ableiten, so hat man nur den Coefficienten C_1 aus der Gleich. (1b) zu berechnen und dann unmittelbar das Gauss-Weierstrass'sche Convergenztheorem anzuwenden, Herr Thomé hat diese Untersuchung mit Anwendung der Fourier'schen Reihen geführt (Journ. f. Math., t.).

EINBECK, 4 August, 1888.

Über die Reduction von Integralen transcendenter Functionen.

Von Leo Koenigsberger.

Bekanntlich hat Abel in dem zweiten Kapitel seines unvollendet gebliebenen "précis d'une théorie des fonctions elliptiques" Untersuchungen über die Form angestellt, welche das Integral eines algebraischen Differentials annehmen muss, wenn man dasselbe durch algebraische, logarithmische und elliptische Transcendenten ausdrückbar voraussetzt, und eine Reihe von Sätzen über die nothwendige rationale Ausführbarkeit solcher Transformationen aufgestellt, welche seitdem Fundamentalsätze der Integralrechnung geworden sind und später Liouville, Tchebichef, Weierstrass, Clebsch u. A. die Möglichkeit boten, auf Grund dieser festen Formen die nothwendigen und hinreichenden Bedingungen für die wirkliche Reduction höherer Integralgattungen auf niedere aufzustellen. Eine Reihe von Untersuchungen aus dem Gebiete der algebraischen Differentialgleichungen haben es jetzt ermöglicht, diese Untersuchungen von der speciellen Klasse der Integrale algebraischer Functionen oder der Abel'schen Integrale auf Integrale beliebiger Transcendenten auszudehnen, und ich erlaube mir im Folgenden, nachdem ich bereits in früheren Arbeiten Andeutungen über die Möglichkeit der Ausdehnung gegeben, diese Probleme, welche zugleich Ergänzungen zu denjenigen Kapiteln der Integralrechnung liefern sollen, welche von den ausführbaren Integralen handeln, an dieser Stelle mit Entwicklung der nothwendigen Sätze aus der Theorie der algebraischen Differentialgleichungen weiter zu verfolgen.

Sei
$$y^{(m)} = f(x, y, y', \ldots . y^{(m-1)}), \tag{1}$$

worin f eine algebraische Function bedeutet, oder in anderer Form,

$$y^{(m)^n} + F_1(x, y, y', \ldots . y^{(m-1)}) y^{(m)^{n-1}} + \ldots . + F_n(x, y, y', \ldots . y^{(m-1)}) = 0 \tag{2}$$

eine algebraische Differentialgleichung m^{ter} Ordnung, und y_1 ein Integral derselben, so soll *die Quadratur*

$$\int y_1 dx = z_1 \tag{3}$$

eine algebraisch ausführbare genannt werden, wenn

$$z_1 = F(x, y_1, y_1', \ldots . y_1^{(m)}, y_2, y_2', \ldots . y_2^{(m)}, y_3, y_3', \ldots . y_3^{(m)}, \ldots .) \tag{4}$$

ist, worin F eine algebraische Function der eingeschlossenen Grössen, $y_1, y_2, y_3, \ldots .$ Integrale der Differentialgleichung (1) *bedeuten, oder wenn z_1 eine Lösung der Gleichung*

$$\begin{aligned} z^\mu + f_1(x, y_1, y_1', \ldots . y_1^{(m)}, y_2, y_2', \ldots . y_2^{(m)}, \ldots .) z^{\mu-1} + \ldots . \\ + f_\mu(x, y_1, y_1', \ldots . y_1^{(m)}, y_2, y_2', \ldots . y_2^{(m)}, \ldots .) = 0 \end{aligned} \tag{5}$$

ist, in welcher $f_1, f_2, \ldots . f_\mu$ rationale Functionen der in den Klammern enthaltenen Grössen darstellen, und die Gleichung (5) *offenbar als eine mit Adjungirung von $x, y_1, y_2, y_3, \ldots .$ und deren Ableitungen algebraisch irreductible angenommen werden darf.*

Ist y_1 eine algebraische Function von x, so soll sie als die Lösung einer algebraischen Differentialgleichung 0^{ter} Ordnung aufgefasst werden.

Zur Auffindung des Grades μ der Gleichung (5) werde zunächst bemerkt, dass die Differentiation von (2)

$$\begin{aligned} &\left(n {y^{(m)}}^{n-1} + (n-1) F_1 {y^{(m)}}^{n-2} + \ldots . + F_{n-1}\right) y^{(m+1)} \\ &\qquad + {y^{(m)}}^{n-1}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_1}{dy} y' + \ldots . + \frac{\partial F_1}{\partial y^{(m-1)}} y^{(m)}\right) + \ldots . \\ &\qquad + \left(\frac{\partial F_n}{\partial x} + \frac{\partial F_n}{\partial y} y' + \ldots . + \frac{\partial F_n}{\partial y^{(m-1)}} y^{(m)}\right) = 0 \end{aligned}$$

also $y^{(m+1)}$ als rationale Function von $x, y, y', \ldots . y^{(m)}$ liefert, und dass somit nach Gleichung (5) durch Differentiation

$$\begin{aligned} &(\mu z^{\mu-1} + (\mu-1) f_1 z^{\mu-2} + \ldots . + f_{\mu-1}) z' \\ &+ z^{\mu-1}\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial y_1} y_1' + \ldots . + \frac{\partial f_1}{\partial y_1^{(m)}} y_1^{(m+1)} + \frac{\partial f_1}{\partial y_2} y_2' + \ldots . + \frac{\partial f_1}{\partial y_2^{(m)}} y_2^{(m+1)} + \ldots .\right) \\ &+ \ldots . + \left(\frac{\partial f_\mu}{\partial x} + \frac{\partial f_\mu}{\partial y_1} y_1' + \ldots .\right) = 0 \end{aligned}$$

oder

$$z' = \frac{\phi_1(x, y_1, y_1', \ldots y_1^{(m)}, y_2, y_2', \ldots y_2^{(m)}, \ldots) z^{\mu-1} + \ldots + \phi_\mu(x, y_1, y_1', \ldots y_1^{(m)}, y_2, y_2', \ldots y_2^{(m)}), \ldots}{\mu z^{\mu-1} + (\mu-1) f_1 z^{\mu-2} + \ldots + f_{\mu-1}} \tag{6}$$

folgt. Setzt man für z das durch (3) definirte Integral z_1 der Differentialgleichung (5), so kann man die in z_1 rationale Function

$$z_1' = \frac{\phi_1 z_1^{\mu-1} + \phi_2 z_1^{\mu-2} + \ldots . + \phi_\mu}{\mu z_1^{\mu-1} + (\mu - 1) f_1 z_1^{\mu-2} + \ldots . + f_{\mu-1}}, \tag{7}$$

wenn $z_2, z_3, \ldots . z_\mu$ die übrigen $\mu - 1$ Lösungen der Gleichung (5) bedeuten, durch Multiplication des Bruches mit dem Werthe des Nenners für diese anderen Lösungen in die Form bringen

$$z_1' = \frac{\begin{array}{r}(\phi_1 z_1^{\mu-1} + \phi_2 z_1^{\mu-2} + \ldots . + \phi_\mu)(\mu z_2^{\mu-1} + (\mu-1) f_1 z_2^{\mu-2} + \ldots . + f_{\mu-1}) \ldots . \\ (\mu z_\mu^{\mu-1} + (\mu-1) f_1 z_\mu^{\mu-2} + \ldots . + f_{\mu-1})\end{array}}{\begin{array}{r}(\mu z_1^{\mu-1} + (\mu-1) f_1 z_1^{\mu-2} + \ldots . + f_{\mu-1})(\mu z_2^{\mu-1} + (\mu-1) f_1 z_2^{\mu-2} + \ldots . + f_{\mu-1}) \ldots . \\ (\mu z_\mu^{\mu-1} + (\mu-1) f_1 z_\mu^{\mu-2} + \ldots . + f_{\mu-1})\end{array}}; \tag{8}$$

da nun der Nenner N eine ganze symmetrische Function von $z_1, z_2, \ldots . z_\mu$ ist, sich also rational und ganz durch die Coefficienten der Gleichung (5) ausdrücken lässt, ferner im Zähler das Product der $\mu - 1$ letzten Factoren eine ganze symmetrische Function von $z_2, z_3, \ldots . z_\mu$ also der Lösungen der aus (5) durch Division mit $z - z_1$ erhaltenen Gleichung

$$z^{\mu-1} + (f_1 + z_1) z^{\mu-2} + (f_2 + f_1 z_1 + z_1^2) z^{\mu-3} + \ldots . + (f_{\mu-1} + f_{\mu-2} z_1 + \ldots . + z_1^{\mu-1}) = 0$$

und sich somit ebenfalls wieder als ganze Function von $z_1, f_1, f_2, \ldots . f_\mu$ darstellen lässt, so wird der gesammte Zähler die Form haben

$$Z = L_0 z_1^r + L_1 z_1^{r-1} + \ldots . + L_r,$$

worin $L_0, L_1, \ldots L_r$ rationale Functionen von $x, y_1, y_2, \ldots$ und deren Ableitungen sind oder auch, da vermöge der Gleichung (5) die höheren Potenzen von z_1 als die $\mu - 1^{\text{te}}$ in gleichartiger Weise linear durch die niederen Potenzen ausgedrückt werden können,

$$Z = M_0 z_1^{\mu-1} + M_1 z_1^{\mu-2} + \ldots . + M_{\mu-1}$$

und somit, da der Nenner N des Bruches (7) den Charakter der Grössen M hatte,

$$z_1' = P_0 z_1^{\mu-1} + P_1 z_1^{\mu-2} + \ldots . + P_{\mu-1}, \tag{9}$$

worin $P_0, P_1, \ldots . P_{\mu-1}$ rationale Functionen von $x, y_1, y_2, \ldots .$ und deren

Ableitungen bis zur m^{ten} Ordnung hin bedeuten. Da aber vermöge der Gleichung (3) $z_1' = y_1$ sein soll, also aus (8)

$$P_0 z_1^{\mu-1} + P_1 z_1^{\mu-2} + \ldots\ldots + P_{\mu-1} = y_1 \tag{10}$$

folgt, so muss, weil die Gleichung (5) als eine mit Adjungirung von x, y_1, $y_2 \ldots\ldots$ und deren Ableitungen bis zur m^{ten} Ordnung hin algebraisch irreductible vorausgesetzt war,

$$P_0 = 0,\ P_1 = 0,\ \ldots\ldots P_{\mu-2} = 0,\ P_{\mu-1} = y_1 \tag{11}$$

sein und somit, da zur Ueberführung des Ausdruckes (6) in (9) z_1 eine willkührliche Lösung der Gleichung (5) sein durfte d. h. die Werthe $P_0, P_1, \ldots\ldots P_{\mu-1}$ für jede andere Lösung dieselben bleiben, so wird sich aus (9) vermöge der gefundenen Beziehungen (11) für jedes der $\alpha = 1, 2, 3, \ldots\ldots \mu$

$$z_\alpha' = y_1 \text{ oder } \int y_1 dx = z_\alpha \tag{12}$$

ergeben. Nun können sich aber die Werthe derselben Quadratur nur um Constanten unterscheiden, und es folgte daher

$$z_\alpha = z_1 + c_\alpha; \tag{13}$$

da jedoch das Zusammenbestehen der aus (5) sich ergebenden Gleichungen

$$z_1^\mu + f_1 z_1^{\mu-1} + f_2 z_1^{\mu-2} + \ldots\ldots + f_{\mu-1} z_1 + f_\mu = 0,$$
$$(z_1 + c_\alpha)^\mu + f_1 (z_1 + c_\alpha)^{\mu-1} + f_2 (z_1 + c_\alpha)^{\mu-2} + \ldots\ldots + f_{\mu-1}(z_1 + c_\alpha) + f_\mu = 0$$

durch Subtraction die Beziehung

$$\mu c_\alpha z_1^{\mu-1} + \left(\frac{\mu(\mu-1)}{2} c_\alpha + (\mu - 1) f_1\right) c_\alpha z_1^{\mu-2} + \ldots\ldots + c_\alpha f_{\mu-1} = 0$$

liefert, welche, weil die Gleichung (5) als irreductibel vorausgesetzt war und die eben erhaltene Gleichung mit jener gleichartig ist, eine identische sein muss, also $c_\alpha = 0$ liefert, dies jedoch nach (13) zwei gleiche Lösungen von (5) voraussetzen würde, was wiederum mit der Annahme der Irreductibilität nicht verträglich ist so folgt, dass die Gleichung (5) überhaupt nur *eine* Lösung haben darf, also linear sein muss und somit z_1 eine rationale Function von x, y_1, y_2, $\ldots\ldots$ und deren Ableitungen ist.

Wir erhalten daher den folgenden Satz:

Wenn die Quadratur $\int y_1 dx,$

worin y_1 ein Integral einer algebraischen Differentialgleichung m^{ter} Ordnung bedeutet, algebraisch ausführbar ist, so lässt sich der Werth derselben stets als RATIONALE *Function von x, particulären Integralen der Differentialgleichung und deren Ableitungen bis einschliesslich zur m^{ten} Ordnung hin darstellen.*

Bemerkt man noch, dass jede rationale Function von $y_1^{(m)}$ sich wieder aus den vorherangegebenen Gründen vermöge der Gleichung (2) als ganze Function $n - 1^{ten}$ Grades von $y_1^{(m)}$ in der Form darstellen lässt

$$\begin{aligned} &\omega_0 \quad (x, y_1, y_1', \ldots . y_1^{(m-1)}, y_2, \ldots . y_2^{(m)}, \ldots .) \, {y_1^{(m)}}^{n-1} \\ + &\omega_1 \quad (x, y_1, y_1', \ldots . y_1^{(m-1)}, y_2, \ldots . y_2^{(m)}, \ldots .) \, {y_1^{(m)}}^{n-2} + \ldots . \\ + &\omega_{n-1} (x, y_1, y_1', \ldots . y_1^{(m-1)}, y_2, \ldots . y_2^{(m)}, \ldots .), \end{aligned}$$

dass wiederum jedes der ω sich wieder in eine ganze Function von $y_2^{(m)}$ umwandeln lässt, u. s. w., so folgt

dass der die Quadratur darstellende, in den Integralen und deren Ableitungen bis zur m^{ten} Ordnung hin rationale Ausdruck sich in eine in den m^{ten} Ableitungen der Integrale ganze Function $n - 1^{ten}$ Grades für jedes derselben umsetzen lässt, deren Coefficienten rationale Functionen von x, den Integralen der Differentialgleichung und deren Ableitungen bis zur $m - 1^{ten}$ Ordnung hin sind.

Bevor wir nun weitere Eigenschaften solcher algebraisch ausführbarer Quadraturen untersuchen, wollen wir eine Hülfuntersuchung vorausschicken, welche sich auf die Form der algebraischen Beziehungen erstreckt, die zwischen den Integralen einer algebraischen Differentialgleichung unter deren Ableitungen stattfinden kann.

Machen wir die Annahme, dass z. B. zwischen drei Integralen und deren Ableitungen für die Differentialgleichung (1) oder (2) eine algebraische Beziehung

$$\Omega(x, y_1, y_1', \ldots . y_1^{(m-1)}, y_2, y_2', \ldots . y_2^{(m-1)}, y_3, y_3' \ldots . . y_3^{(m-1)}) = 0 \quad (14)$$

statt hat, so mag die höchste in (14) vorkommende Ableitung von y_3 die $m - \alpha^{te}$ sein, so dass (14) in die Form gesetzt werden kann

$$y_3^{(m-\alpha)} = f_0(x, y_1, \ldots . y_1^{(m-1)}, y_2, \ldots . y_2^{(m-1)}, y_3, \ldots . y_3^{(m-\alpha-1)}), \quad (15)$$

woraus mit Hülfe von (1) durch Differentiation

$$\begin{aligned} y_3^{(m-\alpha+1)} &= f_1(x, y_1, \ldots . y_1^{(m-1)}, y_2, \ldots . y_2^{(m-1)}, y_3, \ldots . y_3^{(m-\alpha-1)}), \ldots . \\ y_3^{(m)} &= f_\alpha(x, y_1, \ldots . y_1^{(m-1)}, y_2, \ldots . y_2^{(m-1)}, y_3, \ldots . y_3^{(m-\alpha-1)}) \quad (16) \end{aligned}$$

und durch Zusammenstellung mit (1)

$$f(x, y_3, y_3', \dots y_3^{(m-1)}) = f_\alpha(x, y_1, \dots y_1^{(m-1)}, y_2, \dots y_2^{(m-1)}, y_3, \dots y_3^{(m-\alpha-1)}) \quad (17)$$

folgt, welche durch Einsetzen der in den Gleichungen (15) und (16) erhaltenen Werthe von

$$y_3^{(m-\alpha)}, \; y_3^{(m-\alpha+1)}, \dots y_3^{(m-1)}$$

in die linke Seite derselben in

$$\Omega_1(x, y_1, \dots y_1^{(m-1)}, y_2, \dots y_2^{(m-1)}, y_3, \dots y_3^{(m-\alpha-1)}) = 0 \quad (18)$$

übergehen mag. Würden aus dieser Gleichung die Grössen $y_3, y_3', \dots y_3^{(m-\alpha-1)}$ sämmtlich herausfallen, so würde (18) eine algebraische Beziehung nur zwischen y_1, y_2 und deren Ableitungen feststellen—ein Fall, auf den wir das Problem ganz allgemein führen wollen und nachher behandeln—ist dies jedoch nicht der Fall, so mag die höchste in (18) vorkommende Ableitung von y_3 die $m-\beta^{\text{te}}$ sein, wo $\beta > \alpha$, so dass

$$y_3^{(m-\beta)} = F_0(x, y_1, \dots y_1^{(m-1)}, y_2, \dots y_2^{(m-1)}, y_3, \dots y_3^{(m-\beta-1)}) \quad (19)$$

wird und durch Differentiation mit Hülfe von (1) wiederum

$$\begin{aligned} y_3^{(m-\beta+1)} &= F_1\;(x, y_1, \dots y_1^{(m-1)}, y_2, \dots y_2^{(m-1)}, y_3, \dots y_3^{(m-\beta-1)}), \dots \\ y_3^{(m-\alpha)} &= F_{\beta-\alpha}(x, y_1, \dots y_1^{(m-1)}, y_2, \dots y_2^{(m-1)}, y_3, \dots y_3^{(m-\beta-1)}), \end{aligned} \quad (20)$$

woraus durch Zusammenstellung mit (15) und Benutzung der Werthe (19) und (20) sich

$$\Omega_2(x, y_1, \dots y_1^{(m-1)}, y_2, \dots y_2^{(m-1)}, y_3, \dots y_3^{(m-\beta-1)}) = 0 \quad (21)$$

ergiebt. Wäre diese Gleichung nun nicht eine in y_3 und dessen Ableitungen identische, so würde man wieder, wenn die niedrigste Ableitung von y_3 in dieser Gleichung die $m-\gamma^{\text{te}}$ ist, wo $\gamma > \beta$ ist,

$$y_3^{(m-\gamma)} = \Phi_0(x, y_1, \dots y_1^{(m-1)}, y_2, \dots y_2^{(m-1)}, y_3, \dots y_3^{(m-\gamma-1)}) \quad (22)$$

bilden, daraus wiederum mit Hülfe von (1)

$$\begin{aligned} y_3^{(m-\gamma+1)} &= \Phi_1\;(x, y_1, \dots y_1^{(m-1)}, y_2, \dots y_2^{(m-1)}, y_3, \dots y_3^{(m-\gamma-1)}) \dots \\ y_3^{(m-\beta)} &= \Phi_{\gamma-\beta}(x, y_1, \dots y_1^{(m-1)}, y_2, \dots y_2^{(m-1)}, y_3, \dots y_3^{(m-\gamma-1)}) \end{aligned} \quad (23)$$

und durch Zusammenstellung mit (19) mit Benutzung von (22) und (23)

$$\Omega_3(x, y_1, \dots y_1^{(m-1)}, y_2, \dots y_2^{(m-1)}, y_3, \dots y_3^{(m-\gamma-1)}) = 0, \quad (24)$$

und so könnte man fortfahren, bis man, da $\alpha < \beta < \gamma < \dots$ ist, schliesslich auf eine von y_3 und dessen Ableitungen freie algebraische Beziehung zwischen y_1, y_2 und dessen Ableitungen

$$\omega_0(x, y_1, \ldots . y_1^{(m-1)}, y_2, \ldots . y_2^{(m-1)}) = 0, \tag{25}$$

geführt wird; wir nehmen an, dass die durch (24) dargestellte Beziehung bereits die Beziehung (25) war.

Sei nun die in (25) vorkommende niedrigste Ableitung von y_2 die $m - a^{\text{te}}$, so dass sich

$$y_2^{(m-a)} = \psi_0(x, y_1, \ldots . y_1^{(m-1)}, y_2, \ldots . y_2^{(m-a-1)}) \tag{26}$$

ergiebt und daraus

$$\begin{aligned} y_2^{(m-a+1)} &= \psi_1(x, y_1, \ldots . y_1^{(m-1)}, y_2, \ldots . y_2^{(m-a-1)}), \ldots . \\ y_2^{(m)} &= \psi_a(x, y_1, \ldots . y_1^{(m-1)}, y_2, \ldots . y_2^{(m-a-1)}), \end{aligned} \tag{27}$$

so folgt durch Zusammenstellung mit (1)

$$f(x, y_2, y_2', \ldots . y_2^{(m-1)}) = \psi_a(x, y_1, \ldots . y_1^{(m-1)}, y_2, \ldots . y_2^{(m-a-1)}) \tag{28}$$

oder durch Einsetzen der in den Gleichungen (26) und (27) erhaltenen Werthe

$$\omega_1(x, y_1, \ldots . y_1^{(m-1)}, y_2, \ldots . y_2^{(m-a-1)}) = 0; \tag{29}$$

enthält diese Gleichung noch Ableitungen von y_2, und ist die niedrigste die $m - b^{\text{te}}$, wo $b > a$, so setze man

$$y_2^{(m-b)} = \chi_0(x, y_1, \ldots . y_1^{(m-1)}, y_2, \ldots . y_2^{(m-b-1)}), \tag{30}$$

woraus wiederum

$$\begin{aligned} y_2^{(m-b+1)} &= \chi_1 \quad (x, y_1, \ldots . y_1^{(m-1)}, y_2, \ldots . y_2^{(m-b-1)}), \ldots . \\ y_2^{(m-a)} &= \chi_{b-a}(x, y_1, \ldots . y_1^{(m-1)}, y_2, \ldots . y_2^{(m-b-1)}) \end{aligned} \tag{31}$$

und durch Zusammenstellung mit (26) und Benutzung von (30) und (31)

$$\omega_2(x, y_1, \ldots . y_1^{(m-1)}, y_2, \ldots . y_2^{(m-b-1)}) = 0 \tag{32}$$

folgt; fährt man wiederum genau so fort wie oben, so wird man endlich zu einer algebraischen Beziehung

$$\theta(x, y_1, y_1', \ldots . y_1^{(m-1)}) = 0 \tag{33}$$

kommen müssen, welche nur noch y_1 und dessen Ableitungen enthält, und wir nehmen an, dass (32) schon diese Beziehung (33) sei, was ja ohne Beschränkung der Allgemeinheit der Untersuchung geschehen darf. Wir wollen nun sehen, was aus diesen successiven algebraischen Transformationen gefolgert werden kann.

Bisher war über den Charakter der algebraischen Differentialgleichung (1) und der ihr angehörigen Integrale $y_1, y_2, y_3, \ldots.$ noch gar keine Festsetzung getroffen worden. Nehmen wir zuerst an, dass y_1 ein Integral sei, welches nicht schon einer algebraischen Differentialgleichung niederer Ordnung als der m^{ten} genügt, so wird die Gleichung (33) eine identische sein, ihr also auch *jedes* Integral der Differentialgleichung (1) genügen müssen; machen wir jedoch diese Annahme nicht, so wird eine algebraische Differentialgleichung niedrigster Ordnung existiren, von welcher y_1 ein Integral ist und welche die Form haben möge

$$y^{(\lambda)^\nu} + g_1(x, y, y', \ldots. y^{(\lambda-1)})\, y^{(\lambda)^{\nu-1}} + \ldots. + g_\nu(x, y, y', \ldots. y^{(\lambda-1)}) = 0, \tag{34}$$

worin $\lambda \leqq m-1$ und $g_1, g_2, \ldots. g_\nu$ rationale Functionen der eingeschlossenen Grössen sind. Diese Gleichung kann nun als algebraische Gleichung in $y^{(\lambda)}$ aufgefasst mit Adjungirung der Grössen $x, y, y', \ldots. y^{(\lambda-1)}$ irreductibel oder reductibel sein, und im letzteren Falle mag der algebraisch irreductible Factor

$$y^{(\lambda)^\rho} + h_1(x, y, y', \ldots. y^{(\lambda-1)})y^{(\lambda)^{\rho-1}} + \ldots. + h_\rho(x, y, y' \ldots. y^{(\lambda-1)}) = 0 \tag{35}$$

das Integral y_1 zur Lösung haben, so dass die Differentialgleichung (35) und die Differentialgleichung (33), welche wir in der Form

$$y^{(\mu)^\sigma} + k_1(x, y, y', \ldots. y^{(\mu-1)})\, y^{(\mu)^{\sigma-1}} + \ldots. + k_\sigma(x, y, y', \ldots. y^{(\mu-1)}) = 0 \tag{36}$$

darstellen wollen, worin $\mu \leqq m-1$ und $k_1, k_2, \ldots. k_\sigma$ rationale Functionen der eingeschlossenen Grössen sind, eine Lösung y_1 gemeinsam haben. Ist nun $\lambda > \mu$, so wird, da y_1 keiner Differentialgleichung niederer Ordnung als der λ^{ten} genügen sollte, die Differentialgleichung (36) oder (33) identisch sein also auch von jedem Integral von (35) befriedigt werden müssen; ist $\lambda = \mu$, so wird, weil (35) ein algebraisch in Bezug auf $y^{(\lambda)}$ irreductibler Factor war, $\sigma \geqq \rho$ sein müssen, und da dann identisch

$$\left.\begin{aligned} & y^{(\lambda)^\sigma} + k_1 y^{(\lambda)^{\sigma-1}} + \ldots. + k_\sigma \\ = {} & (y^{(\lambda)^\rho} + h_1 y^{(\lambda)^{\rho-1}} + \ldots. + h_\rho)(y^{(\lambda)^{\sigma-\rho}} + l_1 y^{(\lambda)^{\sigma-\rho-1}} + \ldots. + l_{\sigma-\rho}) \\ + {} & n_1 y^{(\lambda)^{\rho-1}} + n_2 y^{(\lambda)^{\rho-2}} + \ldots. + n_{\rho-1} \end{aligned}\right\} \tag{37}$$

ist, so folgt, weil für $y = y_1$ die linken Seiten von (35) und (36) verschwinden, dass auch

$$n_1^{(1)} y_1^{(\lambda)^{\rho-1}} + n_2^{(1)} y_1^{(\lambda)^{\rho-2}} + \ldots. + n_{\rho-1}^{(1)} = 0 \tag{38}$$

ist; da aber die Gleichung (35) eine in $y^{(\lambda)}$ algebraisch irreductible ist, so kann auch nicht eben diese Gleichung für $y = y_1$ also

$$y_1^{(\lambda)^\rho} + h_1(x, y_1, y_1', \dots . y_1^{(\lambda-1)}) y_1^{(\lambda)^{\rho-1}} + \dots . + h_\rho(x, y_1, y_1', \dots . y_1^{(\lambda-1)}) = 0 \quad (39)$$

in Bezug auf $y_1^{(\lambda)}$ mit Adjungirung von $x, y_1, y_1', \dots . y_1^{(\lambda-1)}$ algebraisch reductibel sein; denn wäre dies der Fall, so würde die Differentialgleichung

$$y^{(\lambda)^p} + q_1(x, y, y', \dots . y^{(\lambda-1)}) y^{(\lambda)^{p-1}} + \dots . + q_p(x, y, y_1', \dots . y^{(\lambda-1)}) = 0 \quad (40)$$

mit der Differentialgleichung (35) das Integral y_1 gemein haben, und die Elimination von $y^{(\lambda)}$ zwischen den Gleichungen (35) und (40) eine algebraische Gleichung in $x, y, y', \dots . y^{(\lambda-1)}$ liefern, welche y_1 zum Integral hat, was nach der oben gemachten Annahme, dass y_1 nicht einer Differentialgleichung niederer Ordnung als der λ^{ten} genügen sollte, widerspräche—nur dann würden wir das Eliminationsresultat nicht herleiten können, wenn (40) ein Factor von (35) wäre, was aber auch nicht angeht, weil von (35) angenommen wurde, dass es in Bezug auf $y^{(\lambda)}$ algebraisch irreductibel sei, also muss in allen Fällen auch (39) in Bezug auf $y_1^{(\lambda)}$ algebraisch irreductibel und daher $n_1^{(1)} = n_2^{(1)} = \dots . n_{\rho-1}^{(1)} = 0$; da nun endlich $n_1, n_2, \dots . n_{\rho-1}$ Differentialausdrücke $\lambda - 1^{\text{ter}}$ Ordnung sind, y_1 aber nicht ein Integral einer Differentialgleichung $n - 1^{\text{ter}}$ Ordnung sein sollte, so wird $n_1 = n_2 = \dots . = n_{\rho-1}$ identisch verschwinden müssen, woraus dann nach (37) sich ergeben würde, dass jedes Integral von (35) auch der Gleichung (36) angehört. Ist endlich $\lambda < \mu$, so folgt aus (35) durch Differentiation, wie leicht zu sehen,

$$\left.\begin{array}{l} [\rho y^{(\lambda)^{\rho-1}} + h_1(\rho-1) y^{(\lambda)^{\rho-2}} + \dots . + h_{\rho-1}] y^{(\lambda+1)} + h_1' y^{(\lambda)^{\rho-1}} + \dots . + h_\rho' = 0, \\ [\rho y^{(\lambda)^{\rho-1}} + h_1(\rho-1) y^{(\lambda)^{\rho-2}} + \dots . + h_{\rho-1}]^2 y^{(\lambda+2)} + \Omega_2(x, y, y', \dots . y^{(\lambda)}) = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ [\rho y^{(\lambda)^{\rho-1}} + h_1(\rho-1) y^{(\lambda)^{\rho-2}} + \dots + h_{\rho-1}]^{\mu-\lambda} y^{(\mu)} + \Omega_{\mu-\lambda}(x, y, y', \dots y^{(\lambda)}) = 0, \end{array}\right\} (41)$$

worin $\Omega_2, \Omega_3, \dots . \Omega_{\mu-\lambda}$ ganze Functionen von $y^{(\lambda)}$, deren Grad vermöge der Gleichung (35) auf den $\rho - 1^{\text{ten}}$ reducirt werden kann und deren Coefficienten rationale Functionen von $x, y, y', \dots . y^{(\lambda-1)}$ sind.

Da nun die Gleichung (36) als ganze Function von $y^{(\lambda)}, y^{(\lambda+1)}, y^{(\lambda+2)}, \dots . y^{(\mu)}$ dargestellt werden kann, so wird man dieselbe mit einer so hohen Potenz des Factors $\rho y^{(\lambda)^{\rho-1}} + h_1(\rho-1) y^{(\lambda)^{\rho-2}} + \dots . + h_{\rho-1}$ multipliciren können, dass in der resultirenden Gleichung nur ganze Potenzen der linken Seiten der Gleichungen (41) vorkommen und es wird somit, da die Ω ganze Functionen von $y^{(\lambda)}$

sind und die Potenzen von $y^{(\lambda)}$, welche höher als die $\rho - 1^{\text{te}}$ sind, vermöge der Gleichung (35) durch ganze Functionen niederen Grades ersetzt werden können und zwar für jedes Integral der Gleichung (35),

$$\left.\begin{aligned}&[\rho {y^{(\lambda)}}^{\rho-1} + h_1(\rho-1){y^{(\lambda)}}^{\rho-2} + \ldots . + h_{\rho-1}]^M({y^{(\mu)}}^{\sigma} + k_1 {y^{(\mu)}}^{\sigma-1} + \ldots . + k_\sigma)\\ &\quad = t_0(x, y, y', \ldots . y^{(\lambda-1)}){y^{(\lambda)}}^{\rho-1} + t_1(x, y, y', \ldots . y^{(\lambda-1)}){y^{(\lambda)}}^{\rho-2}\\ &\qquad\qquad + \ldots . + t_{\rho-1}(x, y, y', \ldots . y^{(\lambda-1)})\end{aligned}\right\} \quad (42)$$

sein, worin M eine positive ganze Zahl, und $t_0, t_1, \ldots . t_{\rho-1}$ rationale Functionen der eingeklammerten Grössen bedeuten. Da nun y_1 ein Integral der Differentialgleichung (36) ist und keine der Grössen $h_1, \ldots . h_{\rho-1}$ für $y = y_1$ unendlich werden kann, weil dann der Nenner dieser in $x, y, y', \ldots . y^{(\lambda-1)}$ rationalen Function verschwinden und somit y_1 ein Integral einer Differentialgleichung von niederer Ordnung als der λ^{ten} wäre gegen die Voraussetzung, so wird für $y = y_1$ die linke Seite der Gleichung (42), also auch die rechte verschwinden, und daher wegen der angenommenen algebraischen Irreductibilität der Gleichung (35) in Bezug auf $y^{(\lambda)}$ also auch, wie oben gezeigt worden, in Bezug auf $y_1^{(\lambda)}$,

$$t_0(x, y_1, y_1', \ldots . y^{(\lambda-1)}) = 0, \ldots . t_{\rho-1}(x, y_1, y_1', \ldots . y_1^{(\lambda-1)}) = 0,$$

und daher wiederum, weil y_1 nicht algebraischen Differentialgleichungen $\lambda - 1^{\text{ter}}$ Ordnung genügen sollte,

$$t_0(x, y, y', \ldots . y^{(\lambda-1)}), \ldots . t_{\rho-1}(x, y, y', \ldots . y^{(\lambda-1)})$$

identisch Null und somit

$$[\rho {y^{(\lambda)}}^{\rho-1} + h_1(\rho-1){y^{(\lambda)}}^{\rho-2} + \ldots . + h_{\rho-1}]^M({y^{(\mu)}}^{\sigma} + k_1 {y^{(\mu)}}^{\sigma-1} + \ldots . + k_\sigma) = 0 \quad (43)$$

für alle Integrale der Differentialgleichung (35). Es ergiebt sich somit, dass sämmtliche Integrale der Differentialgleichung (35), welche nicht zugleich der Differentialgleichung

$$\rho {y^{(\lambda)}}^{\rho-1} + h_1(\rho-1){y^{(\lambda)}}^{\rho-2} + \ldots . + h_{\rho-1} = 0$$

genügen,* auch der Differentialgleichung (36) Genüge leisten müssen; nennt man alle Integrale der Differentialgleichung (35), welche zugleich den nach $y^{(\lambda)}$ genommenen Differentialausdruck (44) zu Null machen, also alle Integrale von (35), welche zugleich der Discriminante $\lambda - 1^{\text{ter}}$ Ordnung der Differentialgleichung (35)

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* Nennt man eine algebraische Differentialgleichung eine *irreductible*, die in Bezug auf den höchsten Differentialquotienten mit Adjungirung aller niederen Differentialquotienten algebraisch irreductibel ist und welche die Eigenschaft hat, dass keines ihrer Integrale einer algebraischen Differentialgleichung niederer Ordnung genügt, so wird, wenn die vorgelegte Differentialgleichung (1) also auch (84) irreductibel ist, die Differentialgleichung (44) durch keines der Integrale von (1) befriedigt werden können, und es werden somit *alle* Integrale von (1) der Gleichung (83) Genüge leisten.

$$\begin{vmatrix} 1 & h_1 & h_2 & \dots\, h_\rho & 0 & 0 & \dots & 0 \\ 0 & 1 & h_1 & \dots\, h_{\rho-1} & h_\rho & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 1 \quad h_1 & h_2 & & \dots & h_\rho \\ \rho & (\rho-1)h_1 & & \dots\, h_{\rho-1} & 0 & 0 & \dots & 0 \\ 0 & \rho & (\rho-1)h_1 & \dots & h_{\rho-1} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \rho & (\rho-1)h_1 & \dots & h_{\rho-1} \end{vmatrix} = 0,$$

welche von der $\lambda - 1^{\text{ten}}$ Ordnung ist, befriedigen, *singuläre*, so finden wir also in allen Fällen $\lambda \lesseqgtr \mu$,

dass sämmtliche Integrale der Differentialgleichung (35), *von den singulären abgesehen, auch der Differentialgleichung* (36) *oder* (33) *Genüge leisten.*

Da aber die gegebene Differentialgleichung (1) ebenfalls das Integral y_1 besitzt, also den Charakter der Gleichung (33) hat, so wird aus demselben Grunde jedes Integral von (35), von den singulären abgesehen, auch ein Integral der Differentialgleichung (1) sein, und bezeichnet man nun mit η_1 *irgend ein* nicht singuläres Integral der Differentialgleichung (35), welche als die in Bezug auf den höchsten Differentialquotienten algebraisch irreductible Differentialgleichung niedrigster Ordnung definirt wurde, welche y_1 als Integral besitzt, wobei η_1, wie eben gezeigt worden, auch ein Integral der gegebenen Differentialgleichung (1) sein wird, und bestimmt man eine Function η_2 durch Integration der der Gleichung (30) oder (29) entsprechenden Differentialgleichung

$$\eta_2^{(m-b)} = \chi_0(x, \eta_1, \dots \eta_1^{(m-1)}, \eta_2, \dots \eta_2^{(m-b-1)}), \tag{45}$$

so werden $\eta_2^{(m-b+1)}, \dots \eta_2^{(m-a)}$ durch die entsprechenden Gleichungen (31) dargestellt sein, und da die Gleichung (33) oder (32) nichts anderes war als die Gleichheit der rechten Seiten der letzten Gleichung (31) und der Gleichung (26), so wird also auch (26) entsprechend die Beziehung gelten

$$\eta_2^{(m-a)} = \psi_0(x, \eta_1, \dots \eta_1^{(m-1)}, \eta_2, \dots \eta_2^{(m-a-1)}); \tag{46}$$

da aber aus dieser wiederum die analogen Gleichungen (27) folgen und somit auch, weil (30) erfüllt ist, (28) für η_1, η_2 bestehen muss, aber

$$\eta_2^{(m)} = \psi_a(x, \eta_1, \ldots . \eta_1^{(m-1)}, \eta_2, \ldots . \eta_2^{(m-a-1)}) \tag{47}$$

ist, so muss auch

$$\eta_2^{(m)} = f(x, \eta_2, \ldots . \eta_2^{(m-1)}) \tag{48}$$

also η_2 ein Integral der Differentialgleichung (1) sein und zwar ein zu dem willkührlich gewählten Integral η_1 insofern bestimmt zugehörig als es ein Integral der Differentialgleichung (34) sein muss.

Da nun die Gleichungen (26), also auch (25), worin (24) übergehen sollte, durch η_1 und η_2 befriedigt werden, so wird, wenn man eine Function η_3 als ein Integral der der Gleichung (22) entsprechenden Differentialgleichung

$$\eta_3^{(m-\gamma)} = \phi_0(x, \eta_1, \ldots . \eta_1^{(m-1)}, \eta_2, \ldots . \eta_2^{(m-1)}, \eta_3, \ldots . \eta_3^{(m-\gamma-1)}) \tag{49}$$

bestimmt, auch (23) und somit die der Gleichung (19) entsprechende

$$\eta_3^{(m-\beta)} = F_0(x, \eta_1, \ldots . \eta_1^{(m-1)}, \eta_2, \ldots . \eta_2^{(m-1)}, \eta_3, \ldots . \eta_3^{(m-\beta-1)}) \tag{50}$$

befriedigt sein; da aber danach auch den Gleichungen (20) Genüge geschieht und die letzte der Gleichungen (20) mit (15) zusammengestellt die Gleichung (22), für unseren Fall also (38) liefert, so muss auch (15) entsprechend die Beziehung

$$\eta_3^{(m-a)} = f_0(x, \eta_1, \ldots . \eta_1^{(m-1)}, \eta_2, \ldots . \eta_2^{(m-1)}, \eta_3, \ldots . \eta_3^{(m-a-1)}) \tag{51}$$

bestehen. Da endlich aus (15) die Gleichungen (16) hervorgehen und die Gleichung (17), welche nichts anderes als (18) oder (19), also in unserem Falle (39) ist, erfüllt ist, so folgt

$$\eta_3^{(m)} = f(x, \eta_3, \eta_3', \ldots . \eta_3^{(m-1)}), \tag{52}$$

d. h. auch η_3 ist ein Integral unserer Differentialgleichung (1); nun ist aber die Beziehung (40) der Gleichung (15) analog nichts anderes als die der Gleichung (14) entsprechende Beziehung

$$\Omega(x, \eta_1, \ldots . \eta_1^{(m-1)}, \eta_2, \ldots . \eta_2^{(m-1)}, \eta_3, \ldots . \eta_3^{(m-1)}) = 0, \tag{53}$$

in welcher η_1 ein willkührliches Integral der Differentialgleichung (35) und somit auch ein Integral der Differentialgleichung (1), η_2 und η_3 die Gleichungen (34) und (38) befriedigende Integrale derselben Differentialgleichung (1) bedeuten.

Wir erhalten somit den nachfolgenden Satz:

Besteht zwischen Integralen und deren Ableitungen für eine algebraische Differentialgleichung beliebiger Ordnung eine algebraische Beziehung, so bleibt diese erhalten, wenn statt eines dieser Integrale jedes andere dieser Differentialgleichung gesetzt wird, welches ein beliebiges nicht singuläres Integral der in Bezug auf den höchsten Differentialquotienten algebraisch irreductibeln Differentialgleichung niedrigster Ordnung ist, welcher jenes Integral Genüge leistet, während für die anderen in der Beziehung vorkommenden Integrale der Differentialgleichung (1) *bestimmte andere zu substituiren sind.*

Nehmen wir nun an, die gegebene Differentialgleichung (1) sei selbst in Bezug auf die höchste Ableitung algebraisch irreductibel und eines der in der gegebenen algebraischen Relation zwischen den Integralen und deren Ableitungen vorkommenden Integrale genüge keiner Differentialgleichung niederer Ordnung, so wird der Satz gelten,

dass unter dieser für die Differentialgleichung (1) *gemachten Annahme dieses eine Integral durch ein willkührliches nicht singuläres Integral der Differentialgleichung* (1) *ersetzt werden darf, wenn nur für die anderen in der algebraischen Beziehung vorkommenden Integrale passende Integrale eben dieser Differentialgleichung substituirt werden.*

Nach Feststellung dieser Sätze gehe ich wieder zur Untersuchung der algebraisch ausführbaren Quadraturen über, von denen wir bewiesen hatten, dass dieselben sich in die Form

$$\int y_1 dx = F(x, y_1, y_1', \ldots . y_1^{(m)}, y_2, y_2', \ldots . y_2^{(m)}, y_3, y_3', \ldots . y_3^{(m)}, \ldots .) \tag{54}$$

setzen lassen, wenn $y_1, y_2, y_3, \ldots .$ Integrale der Differentialgleichung m^{ter} Ordnung (1) und F eine ganze Function von $y_1^{(m)}, y_2^{(m)}, y_3^{(m)}, \ldots .$ darstellt, deren Coefficienten rationale Functionen von $x, y_1, y_2, y_3, \ldots .$ und den Ableitungen dieser Grössen bis zur $m-1^{\text{ten}}$ Ordnung hin sind. Da nun die Differentiation von (54) zwischen Integralen der Differentialgleichung (1) und deren Ableitungen die algebraische Beziehung

$$y_1 = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y_1} y_1' + \ldots + \frac{\partial F}{\partial y_1^{(m)}} y_1^{(m+1)} + \frac{\partial F}{\partial y_2} y_2' + \ldots + \frac{\partial F}{\partial y_2^{(m)}} y_2^{(m+1)} + \ldots \tag{55}$$

liefert, diese aber nach dem vorher erhaltenen Satze bestehen bleibt, wenn statt y_1 ein jedes nicht singuläre Integral η_1 der Differentialgleichung (35) also auch von

(1) gesetzt wird, während $y_2, y_3, \ldots.$ durch passende Integrale $\eta_2, \eta_3, \ldots.$ der Differentialgleichung (1) zu ersetzen sind, so folgt also

$$\eta_1 = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial \eta_1}\eta_1' + \ldots + \frac{\partial F}{\partial \eta_1^{(m)}}\eta_1^{(m+1)} + \frac{\partial F}{\partial \eta_2}\eta_2' + \ldots + \frac{\partial F}{\partial \eta_2^{(m)}}\eta_2^{(m+1)} + \ldots \tag{56}$$

oder

$$\int \eta_1 dx = F(x, \eta_1, \eta_1', \ldots. \eta_1^{(m)}, \eta_2, \eta_2', \ldots. \eta_2^{(m)}, \eta_3, \eta_3', \ldots. \eta_3^{(m)}, \ldots.) \tag{57}$$

und es ergiebt sich somit der Satz:

Wenn die Quadratur eines Integrales einer algebraischen Differentialgleichung m^{ter} Ordnung algebraisch ausführbar ist, so ist es auch die Quadratur eines jeden anderen Integrales eben dieser Differentialgleichung, welches ein beliebiges nicht singuläres Integral der in Bezug auf den höchsten Differentialquotienten algebraisch irreductibeln Differentialgleichung niedrigster Ordnung ist, welcher jenes Integral Genüge leistet, und zwar in Form DERSELBEN *ganzen Function der m^{ten} Ableitungen von Integralen der gegebenen Differentialgleichung m^{ter} Ordnung mit Coefficienten, welche rationale Functionen der unabhängigen Variabeln, der vorkommenden Integrale und deren Ableitungen bis zur $m - 1^{ten}$ Ordnung hin sind.*

Ist die gegebene Differentialgleichung m^{ter} Ordnung wieder selbst in Bezug auf die höchste Ableitung irreductibel, und genügt das die Basis der algebraisch ausführbaren Quadratur bildende Integral nicht schon einer algebraischen Differentialgleichung niederer Ordnung, so folgt

dass die rationale Form der algebraisch ausführbaren Quadratur erhalten bleibt, wenn die Basis derselben durch ein BELIEBIGES *nicht singuläres Integral der Differentialgleichung ersetzt wird, während für die anderen in der rationalen Function F vorkommenden Integrale dieser Differentialgleichung passende Integrale substituirt werden.*

Betrachten wir den speciellen Fall, dass die algebraisch ausführbare Quadratur nur die Basis der Quadratur und deren Ableitungen enthält, so dass

$$\int y_1 dx = \omega_0(x, y_1, y_1', \ldots. y_1^{(m-\alpha)}) \tag{58}$$

ist, worin ω_0 eine algebraische Function und $1 \leqq \alpha \leqq m$ ist, so wird, weil

$$y_1 = \frac{\partial \omega_0}{\partial x} + \frac{\partial \omega_0}{\partial y_1}y_1' + \frac{\partial \omega_0}{\partial y_1'}y_1'' + \ldots. + \frac{\partial \omega_0}{\partial y_1^{(m-1)}}y_1^{(m-\alpha+1)}) \tag{59}$$

ist, y_1 jedenfalls auch schon einer algebraischen Differentialgleichung $m - \alpha + 1^{ter}$

Ordnung genügen, wenn also $\alpha > 1$ ist, einer algebraischen Differentialgleichung von niederer Ordnung als der m^{ten} und wir sehen somit,

dass, wenn y_1 das Integral einer Differentialgleichung m^{ter} Ordnung ist, welches nicht schon einer algebraischen Differentialgleichung niederer Ordnung Genüge leistet, eine algebraisch ausführbare Quadratur nothwendig die $m - 1^{\text{te}}$ Ableitung der Basis enthalten muss.

Dasselbe gilt also für jede algebraisch ausführbare Quadratur irgend eines Integrales einer irreductibeln Differentialgleichung.

Soll die Quadratur irgend einer transcendenten Function y_1 sich algebraisch durch eben diese und deren Ableitungen ausdrücken lassen, so muss diese das Integral einer algebraischen Differentialgleichung sein,

wie aus der Gleichung (58), die der Annahme nach stattfinden würde, hervorgeht, und jedenfalls ein Integral einer algebraischen Differentialgleichung $m - \alpha + 1^{\text{ter}}$ Ordnung.

Die Frage also, welche transcendente Functionen mit Hülfe eben dieser Transcendenten algebraisch ausführbare Quadraturen besitzen, würde vermöge

$$\int y_1 dx = F(x, y_1) \tag{60}$$

also auf die Differentialgleichung erster Ordnung

$$y = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} y' \text{ oder } y'^n + f_1(x, y) y'^{n-1} + \dots + f_n(x, y) = 0 \tag{61}$$

führen, vermöge welcher wiederum nach den oben bewiesenen Sätzen die Gleichung (60) sich in

$$\int y_1 dx = \omega_0(x, y_1) + \omega_1(x, y_1) y_1' + \dots + \omega_n(x, y_1) y_1'^{n-1} \tag{62}$$

transformiren lassen müsste,* worin $\omega_0, \omega_1, \dots \omega_n$ rationale Functionen der Grössen x und y_1 sind; soll also y_1 das Integral einer Differentialgleichung sein, welches nicht schon einer Differentialgleichung niederer Ordnung genügt, soll also

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* Der von Abel bewiesene Satz, dass, wenn das Integral einer algebraischen Function selbst algebraisch ist, der Werth desselben sich rational durch x und die Basis der Quadratur darstellen lasse, folgt aus dem Obigen offenbar für den speciellen Fall der Differentialgleichung 0^{ter} Ordnung, so dass für eine algebraische Function y_1

$$\int y_1 dx = \omega_0(x) + \omega_1(x) y_1 + \omega_2(x) y_1^2 + \dots + \omega_{n-1}(x) y_1^{n-1}$$

wird.

z. B. y_1 das Integral einer irreductibeln Differentialgleichung sein, so muss dieselbe von der ersten Ordnung sein.

Ist die Differentialgleichung 1ter Ordnung, von welcher y_1 ein Integral ist, von der Form
$$y' = f(x, y), \tag{63}$$
worin f eine rationale Function bedeutet, so wird die Gleichung (62) in
$$\int y_1 dx = \omega_0(x, y_1) \tag{64}$$
übergehen, worin ω_0 wiederum rational ist, und wir werden daher den mit dem Abel'schen Satze für Integrale algebraischer Functionen, welcher in der Anmerkung angeführt wurde, gleichlautenden Satz erhalten:

Wenn die Quadratur eines Integrales einer Differentialgleichung erster Ordnung, welche die Ableitung als eine rationale Function der unabhängigen und abhängigen Variabeln darstellt, algebraisch ausführbar ist, so lässt sich der Werth derselben stets als rationale Function des Integrales und der unabhängigen Variabeln ausdrücken.

Man sieht zugleich aus (64), dass man sich alle diese Differentialgleichungen erster Ordnung wirklich herstellen kann, indem man in
$$\frac{\partial\omega_0}{\partial x} - y + \frac{\partial\omega_0}{\partial y} y' = 0 \tag{65}$$
ω_0 eine beliebige rationale Function von x und y bedeuten lässt.

Will man nur die *linearen* Differentialgleichungen erster Ordnung aufstellen, für welche die Quadratur eines Integrales algebraisch also nach dem Obigen auch rational durch dieses Integral ausdrückbar ist, so wird nach dem obigen Satze die Relation (64) bestehen bleiben, wenn man statt y_1 irgend ein anderes Integral der linearen Differentialgleichung
$$y' = yf_1(x) + f_2(x) \tag{66}$$
welche singuläre Integrale überhaupt nicht hat, also auch
$$y = y_1 + ce^{\int f_1(x)dx}, \tag{67}$$
worin c eine willkührliche Constante bedeutet; dann geht aber (64) über in
$$\int \left(y_1 + ce^{\int f_1(x)dx}\right) dx = \omega_0\left(x, y_1 + ce^{\int f_1(x)dx}\right)$$

oder vermöge (64) in

$$\omega_0\left(x,\, y_1 + ce^{\int f_1(x)\,dx}\right) = \omega_0(x,\, y_1) + c\int e^{\int f_1(x)\,dx}\,dx. \tag{68}$$

Da diese Gleichung für jedes c bestehen muss, so folgt aus der Differentiation nach dieser Grösse

$$\frac{\partial\omega_0\left(x,\, y_1 + ce^{\int f_1(x)\,dx}\right)}{\partial\left(y_1 + ce^{\int f_1(x)\,dx}\right)} = e^{-\int f_1(x)\,dx}\int e^{\int f_1(x)\,dx}\,dx$$

also der Differentialquotient von ω_0 nach dem ganzen zweiten Argument genommen von c also auch vom zweiten Argumente selbst unabhängig und somit

$$\omega_0(x,\, y_1) = P_1 y_1 + P_2, \tag{69}$$

worin P_1 und P_2 rationale Functionen von x sind; setzt man diesen Werth von ω_0 in (68) ein, so folgt

$$P_1\left(y_1 + ce^{\int f_1(x)\,dx}\right) + P_2 = P_1 y_1 + P_2 + c\int e^{\int f_1(x)\,dx}\,dx$$

und somit der Werth von P_1 in der Form

$$P_1 = e^{-\int f_1(x)\,dx}\int e^{\int f_1(x)\,dx}\,dx, \tag{70}$$

welcher eine rationale Function von x sein muss, oder, was dasselbe ist, es muss die lineare Differentialgleichung erster Ordnung

$$y' + f_1(x)\,y = 1 \tag{71}$$

ein rationales Integral und zwar P_1 haben. Da aber für den in (69) gefundenen Werth von ω_0 sich aus (65) die Differentialbeziehung ergiebt

$$P_1'y + P_2' - y + P_1 y' = 0$$

oder nach (66)

$$P_1'y + P_2' - y + P_1\left(yf_1(x) + f_2(x)\right) = 0,$$

so muss, weil P_1 und P_2 rationale Functionen von x, und y nicht algebraisch sein sollte,

$$P_1' + P_1 f_1(x) = 1 \quad \text{und} \quad P_2' = -P_1 f_2(x) \tag{72}$$

sein, wovon die erste Gleichung die durch (71) ausgedrückte Bedingung liefert, während die zweite aussagt, dass

$$\int dx f_2(x)\, e^{-\int f_1(x)\,dx}\int e^{\int f_1(x)\,dx}\,dx = -P_2$$

eine rationale Function ist.

Sind aber umgekehrt

$$e^{-\int f_1(x)\,dx}\int e^{\int f_1(x)\,dx}dx = P_1 \text{ und } \int P_1 f_2(x)\,dx = -P_2 \qquad (73)$$

rationale Functionen, so folgt, dass, weil

$$y_1 = e^{\int f_1(x)\,dx}\int e^{-\int f_1(x)\,dx} f_2(x)\,dx \qquad (74)$$

ein Integral der Differentialgleichung (66) ist, durch partielle Integration

$$\int y_1 dx = \int e^{\int f_1(x)\,dx}dx \times \int e^{-\int f_1(x)\,dx} f_2(x)\,dx$$

$$-\int\left[e^{-\int f_1(x)\,dx} f_2(x)\int e^{\int f_1(x)\,dx}dx\right]dx = P_1 y_1 + P_2,$$

und wir erhalten somit den folgenden Satz:

Die nothwendigen und hinreichenden Bedingungen dafür, dass eine lineare Differentialgleichung erster Ordnung

$$y' = f_1(x)\,y + f_2(x)$$

die Eigenschaft besitzt, dass die Quadratur über eines ihrer Integrale algebraisch durch eben dieses Integral ausdrückbar ist, sind die, dass

$$e^{-\int f_1(x)\,dx}\int e^{\int f_1(x)\,dx}dx = P_1 \text{ und } \int P_1 f_2(x)\,dx = -P_2$$

rationale Functionen von x sind, und die Reductionsformel der Quadratur lautet dann

$$\int y\,dx = P_1 y + P_2;$$

es folgt zugleich aus (72), *dass $f_1(x)$ und $f_2(x)$ rationale Functionen von x sein müssen.*

Will man solche Fälle herleiten, so braucht man nur in der Differentialgleichung

$$y' + y f_1(x) = 1$$

für y irgend eine rationale Function von x zu substituiren, so folgt $f_1(x)$ als rationale Function, während das substituirte y die Grösse P_1 darstellte und dann ist $f_2(x)$ so zu wählen, dass P_2 wieder rational wird, oder $f_2(x) = -\frac{P_2'}{P_1}$, so dass man für P_2 eine willkührliche rationale Function wählen und so $f_2(x)$ bestimmen kann.

Wir wollen nun aber allgemein nach den nothwendigen und hinreichenden Bedingungen fragen, unter denen die Quadratur über das Integral einer linearen Differentialgleichung m^{ter} Ordnung

$$y^{(m)} + f_1(x)\, y^{(m-1)} + \ldots . + f_m(x)\, y = f(x), \tag{75}$$

in welcher $f_1(x), f_2(x) \ldots . f_m(x), f(x)$ rationale Functionen von x bedeuten, wieder eine algebraische Function dieses Integrales selbst ist.

Sei also y_1 das Integral der Differentialgleichung (75) und

$$\int y_1 dx = F(x, y_1), \tag{76}$$

worin F eine algebraische Function bedeutet, so werden, wenn wir aus der Differentialgleichung

$$y = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\, y' \tag{77}$$

den in y' algebraisch irreductibeln Factor absondern, welcher die transcendente Function y_1 zum Integral hat und welcher

$$y'^{n} + \phi_1(x, y)\, y'^{n-1} + \ldots . + \phi_n(x, y) = 0 \tag{78}$$

sein möge, worin $\phi_1, \phi_2, \ldots . \phi_n$ rationale Functionen bedeuten, nach dem oben bewiesenen Hülfsatze, nach welchem, wenn eine im höchsten Differentialquotienten algebraisch irreductible Differentialgleichung ein Integral, das nicht schon einer Differentialgleichung niederer Ordnung angehört, mit einer anderen Differentialgleichung gemein hat, jedes nicht singuläre Integral der ersteren auch ein Integral der letzteren sein muss, alle nicht singulären Integrale von (78) auch Integrale der linearen Differentialgleichung (75) sein müssen. Da nun das allgemeine Integral von (75) die Form hat

$$y = c_1\eta_1 + c_2\eta_2 + \ldots . + c_m\eta_m + L,$$

wenn $\eta_1, \eta_2, \ldots . \eta_m$ ein Fundamentalsystem von Integralen der reducirten Differentialgleichung

$$y^{(m)} + f_1(x)\, y^{(m-1)} + \ldots . + f_m(x)\, y = 0 \tag{80}$$

ist, und

$$L = \eta_1 \int \frac{\Delta_1}{\Delta} f(x)\, dx + \eta_2 \int \frac{\Delta_2}{\Delta} f(x)\, dx + \ldots . + \eta_m \int \frac{\Delta_m}{\Delta} f(x)\, dx, \tag{81}$$

worin

$$\Delta = \begin{vmatrix} \eta_1 & \eta_2 & \ldots . \eta_m \\ \eta_1' & \eta_2' & \ldots . \eta_m' \\ \ldots & \ldots & \ldots \\ \eta_1^{(m-1)} & \eta_2^{(m-1)} & \ldots . \eta_m^{(m-1)} \end{vmatrix}, \quad \Delta_r = \begin{vmatrix} \eta_1 & \ldots . \eta_{r-1} & \eta_{r+1} & \ldots . \eta_m \\ \eta_1' & \ldots . \eta_{r-1}' & \eta_{r+1}' & \ldots . \eta_m' \\ \ldots & \ldots & \ldots & \ldots \\ \eta_1^{(m-2)} & \ldots . \eta_{r-1}^{(m-2)} & \eta_{r+1}^{(m-2)} & \ldots . \eta_m^{(m-2)} \end{vmatrix} \tag{82}$$

ist, so muss auch das allgemeine Integral von (78), da es ein Integral von (75) sein muss, die Form haben

$$y = R_1\eta_1 + R_2\eta_2 + \dots + R_m\eta_m + L, \tag{83}$$

worin $R_1, R_2, \dots R_m$ Functionen *einer* willkührlichen Constanten sein werden. Greifen wir $m + 1$ particuläre Integrale von (78) heraus, so dass

$$\left.\begin{array}{llllll} y_1 & = R_{11}\eta_1 & + R_{12}\eta_2 & + \dots + R_{1m}\eta_m & + L, \\ y_2 & = R_{21}\eta_1 & + R_{22}\eta_2 & + \dots + R_{2m}\eta_m & + L, \\ \dots & \dots & \dots & \dots & \dots \\ y_{m+1} & = R_{m+11}\eta_1 & + R_{m+12}\eta_2 & + \dots + R_{m+1m}\eta_m & + L \end{array}\right\} \tag{84}$$

wird, so wird die Determinante des linearen Systems (83) und (84)

$$(D)\quad \begin{vmatrix} y & R_1 & R_2 & \dots R_m & 1 \\ y_1 & R_{11} & R_{12} & \dots R_{1m} & 1 \\ y_2 & R_{21} & R_{22} & \dots R_{2m} & 1 \\ \dots & \dots & \dots & \dots & \dots \\ y_{m+1} & R_{m+11} & R_{m+12} & \dots R_{m+1m} & 1 \end{vmatrix} = 0$$

sein müssen, und somit das allgemeine Integral der Differentialgleichung erster Ordnung (78) sich als homogene lineare Function von $m + 1$ particulären Integralen in der Form

$$S_0 y + S_1 y_1 + S_2 y_2 + \dots + S_{m+1} y_{m+1} = 0 \tag{85}$$

ergeben; ist die lineare Differentialgleichung (75) homogen, so wird $L = 0$ und somit das allgemeine Integral von (78) sich als homogene lineare Function von m particulären Integralen ergeben. Nur dann wird diese Relation nicht existiren, wenn $S_0, S_1, \dots S_{m+1}$ sämmtlich verschwinden würden, und man könnte dann andere $m + 1$ particuläre Integrale wählen, wodurch auch die von der Integrationsconstanten abhängigen Werthe der R also auch der S sich ändern; aber es könnte sein, dass die S Grössen für eine beliebige Wahl der particulären Integrale verschwinden und es fragt sich, was dies für die Grössen $y, y_1, \dots y_{m+1}$ aussagt. Stellt man in diesem Falle die ersten $m + 1$ Gleichungen zusammen

$$\left.\begin{array}{l} y = R_1\eta_1 + R_2\eta_2 + \dots + R_m\eta_m + L.1, \\ y_1 = R_{11}\eta_1 + R_{12}\eta_2 + \dots + R_{1m}\eta_m + L.1, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ y_m = R_{m1}\eta_1 + R_{m2}\eta_2 + \dots + R_{mm}\eta_m + L.1, \end{array}\right\}, \tag{86}$$

so wird der Annahme nach die Determinante der rechten Seite verschwinden und somit wenn man die Gleichungen mit den nach der ersten Verticalreihe genommenen Unterdeterminanten m^{ter} Ordnung T_0, $T_1, \dots \, T_m$ multiplicirt, nach bekannten Determinanteneigenschaften eine lineare homogene Beziehung von der Form

$$T_0 y + T_1 y_1 + \dots + T_m y_m = 0 \tag{87}$$

existiren, sich also das allgemeine Integral als homogene lineare Function von m particulären Integralen ergeben; sind auch diese Unterdeterminanten sämmtlich Null, so nehme man die m Gleichungen

$$\left.\begin{array}{ll} y & = R_1\eta_1 + R_2\eta_2 + \dots + R_m\eta_m + L.1, \\ y_1 & = R_{11}\eta_1 + R_{12}\eta_2 + \dots + R_{1m}\eta_m + L.1, \\ \multicolumn{2}{l}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\ y_{m-1} & = R_{m-11}\eta_1 + R_{m-12}\eta_2 + \dots + R_{m-1m}\eta_m + L.1, \end{array}\right\} \tag{88}$$

und multiplicire die Gleichungen wieder z. B. mit den nach der ersten Verticalreihe genommenen Unterdeterminanten $m - 1^{\text{ter}}$ Ordnung U_0, $U_1, \dots \, U_{m-1}$ der Determinante der R, so werden wiederum nach der Annahme, dass *alle* Unterdeterminanten m^{ter} Ordnung der Determinante (D) verschwinden, wie unmittelbar zu sehen,

$$U_0 y + U_1 y_1 + U_2 y_2 + \dots + U_{m-1} y_{m-1} = 0 \tag{89}$$

sein u. s. w. Da endlich nicht alle Unterdeterminanten 2^{ter} Ordnung verschwinden können, so bleibt unter allen Umständen der Satz bestehen,

dass, wenn die Differentialgleichung (78) *erster Ordnung** *mit der linearen Differentialgleichung* m^{ter} *Ordnung ein Integral gemeinsam hat, das allgemeine Integral der ersteren eine lineare homogene Function von* $m + 1$ *ihrer particulären Integrale sein wird, wobei die Coefficienten einzelner dieser Integrale verschwinden können*:†

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* Da bei dem Beweise dieses Satzes die Ordnung der Differentialgleichung (78) nicht in Frage kam, so bleibt er bestehen für jede algebraische Differentialgleichung, welche mit einer linearen Differentialgleichung ein Integral gemeinsam hat.

† So hat z. B. die lineare Differentialgleichung zweiter Ordnung

$$y'' - 3y' + 2y = 0 \tag{a}$$

mit der Differentialgleichung erster Ordnung

$$y'^2 - (4y + 1)\,y' + y\,(4y + 1) = 0 \tag{b}$$

das Integral

$$y_1 = e^x + e^{2x} \tag{c}$$

und es folgt daher aus dem Vorigen,

dass, wenn eine lineare Differentialgleichung m^{ter} Ordnung die Eigenschaft hat, dass die Quadratur über eines ihrer Integrale algebraisch durch eben dieses Integral ausdrückbar ist, dieses Integral einer solchen in Bezug auf den ersten Differentialquotienten algebraisch irreductibeln Differentialgleichung erster Ordnung genügen muss, deren allgemeines Integral eine homogene lineare Function von $m+1$ particulären Integralen derselben ist, deren Coefficienten Functionen EINER *willkührlichen Constanten sind.*

Nachdem die Differentialgleichung (78), welche mit der vorgelegten linearen Differentialgleichung das Integral y_1 gemein hat, durch die in dem eben bewiesenen Satze ausgesprochene Eigenschaft charakterisirt worden, bleibt die Form derjenigen im ersten Differentialquotienten algebraisch irreductibeln Differentialgleichungen erster Ordnung zu untersuchen übrig, für welche das allgemeine Integral eine lineare homogene Function mit constanten Coefficienten von particulären Integralen eben derselben Differentialgleichung ist.

Sei also für eine algebraische Differentialgleichung erster Ordnung

$$y' = f(x, y), \tag{90}$$

welche in Bezug auf y' algebraisch irreductibel sein mag; das allgemeine Integral

$$y = M_1 y_1 + M_2 y_2 + \ldots + M_r y_r, \tag{91}$$

worin $y_1, y_2, \ldots y_r$ particuläre Integrale von (91) und $M_1, M_2, \ldots M_r$ Functionen einer willkührlichen Constanten sein sollen, so folgt durch Differentiation der Beziehung (91) mit Benutzung von (90) die Relation

$$M_1 f(x, y_1) + M_2 f(x, y_2) + \ldots + M_r f(x, y_r) = f(x, M_1 y_1 + M_2 y_2 + \ldots + M_r y_r), \tag{92}$$

also entweder eine identische Gleichung, oder eine algebraische Beziehung zwischen den r particulären Integralen.

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gemein; nun ist aber das allgemeine Integral von (b) in der Form enthalten

(d) $$y = ce^x + c^2 e^{2x}$$

und es besteht somit, wenn das dem Werthe c entsprechende particuläre Integral mit y_2 bezeichnet wird, also

(e) $$y_2 = c_2 e^x + c_2^2 e^{2x}$$

ist, vermöge der Gleichungen (c), (d), (e) zwischen dem allgemeinen Integrale y und den beiden particulären Integralen y_1 und y_2 die homogene lineare Beziehung

$$\begin{vmatrix} y & c & c^2 \\ y_1 & 1 & 1 \\ y_2 & c_2 & c_2^2 \end{vmatrix} = 0.$$

Sei die Gleichung (92) zunächst eine identische, so muss sie bestehen bleiben, wenn dieselbe sowohl einzeln nach $y_1, y_2, \ldots . y_r$ differentiirt wird, und man erhält somit

$$\frac{\partial f(x, M_1y_1 + M_2y_2 + \ldots . + M_ry_r)}{\partial(M_1y_1 + M_2y_2 + \ldots . + M_ry_r)} = \frac{\partial f(x, y_\rho)}{\partial y_\rho} = \frac{\partial f(x, y_\sigma)}{\partial y_\sigma}, \tag{93}$$

es hängt somit der Differentialquotient der Function $f(x, y)$ nicht mehr von y ab und es ist daher

$$f(x, y) = f_1(x)\,y + f_2(x),$$

so dass die Differentialgleichung erster Ordnung (90) in die lineare übergeht

$$y' = f_1(x)\,y + f_2(x); \tag{94}$$

zugleich sieht man, dass für (94) die Beziehung zwischen dem allgemeinen und den particulären Integralen in die einfache Form gesetzt werden kann

$$y = \frac{c_2 - c}{c_2}\,y_1 + \frac{c}{c_2}\,y_2,$$

worin c eine willkührliche, c_2 eine specielle Integrationsconstante bedeuten.

Es bleibt jedoch nun noch der schwierigere Fall zu untersuchen, in dem die Gleichung (92) keine identische ist, sondern eine algebraische Beziehung zwischen den r particulären Integralen der Differentialgleichung (90) liefert, und wir wollen hier diese Untersuchung für den Fall durchführen, dass die lineare Differentialgleichung (75) homogen von der zweiten Ordnung ist, also lautet

$$y'' + f_1(x)\,y' + f_2(x)\,y = 0, \tag{95}$$

sich also für die Differentialgleichung erster Ordnung (78) das allgemeine Integral als homogene lineare Function von 2 particulären Integralen in der Form

$$y = M_1y_1 + M_2y_2 \tag{96}$$

ergiebt, für welche die vorausgesetzte algebraische Beziehung zwischen den particulären Integralen

$$y_2 = F(x, y_1) \tag{97}$$

lauten möge. Da nun nach dem oben bewiesenen Satze aus der Beziehung (97) vermöge (96) die Relation

$$N_1y_1 + N_2y_2 = F(x, M_1y_1 + M_2y_2), \tag{98}$$

hervorgeht, worin M_1, M_2, N_1, N_2 feste Functionen einer willkührlichen Constanten c sind, ausserdem (98) vermöge (97) in

$$N_1 y_1 + N_2 F(x, y_1) = F(x, M_1 y_1 + M_2 F(x, y_1)) \tag{99}$$

übergeht, und diese Beziehung endlich, da y_1 keine algebraische Function von x sein sollte, eine in x, y_1, c identische sein muss, so erhält man durch Differentiation von (99) nach y_1 und c

$$N_1 + N_2 \frac{\partial F(x, y_1)}{\partial y_1} = \frac{\partial F(x, M_1 y_1 + M_2 F(x, y_1))}{\partial (M_1 y_1 + M_2 F(x, y_1)} \left(M_1 + M_2 \frac{\partial F(x, y_1)}{\partial y_1} \right)$$

und

$$N_1' y_1 + N_2' F(x, y_1) = \frac{\partial F(x, M_1 y_1 + M_2 F(x, y_1))}{\partial (M_1 y_1 + M_2 F(x, y_1))} (y_1 M_1' + M_2' F(x, y_1))$$

und hieraus

$$AF(x, y_1) \frac{\partial F(x, y_1)}{\partial y_1} + By_1 \frac{\partial F(x, y_1)}{\partial y_1} + CF(x, y_1) + Dy_1 = 0, \tag{100}$$

worin A, B, C, D Constanten bedeuten, und die wiederum eine identische sein muss.

Setzt man hierin

$$F(x, y_1) = Y_1, \tag{101}$$

so geht (100) in die Differentialgleichung

$$(AY_1 + By_1)\, dY_1 + (CY_1 + Dy_1)\, dy_1 = 0, \tag{102}$$

über, oder wenn

$$Y_1 = y_1 u \tag{103}$$

gesetz wird, in

$$(Au + B)(y_1 du + u dy_1) + (Cu + D)\, dy_1 = 0, \tag{104}$$

deren Integral durch

$$\log y_1 = -\int \frac{Au + B}{Au^2 + (B + C)u + D}\, du + \log k \tag{105}$$

dargestellt, worin k von x abhängen kann. Nun ist aber

$$\frac{Au + B}{Au^2 + (B + C)u + D} = \frac{A\alpha + B}{2A\alpha + B + C} \frac{1}{u - \alpha} + \frac{A\beta + B}{2A\beta + B + C} \frac{1}{u - \beta} = \frac{\lambda}{u - \alpha} + \frac{1 - \lambda}{u - \beta},$$

worin λ den Coefficienten des ersten Partialbruches bedeudet, und daher nach (105)

$$k y_1^{-1} = (u - \alpha)^{\lambda} (u - \beta)^{1 - \lambda}, \tag{106}$$

worin λ, weil u eine algebraische Function von y_1 sein muss, eine rationale Zahl sein wird. Aus den Gleichungen (97), (101) und (103) ergiebt sich somit

$$\left(\frac{y_2 - \alpha y_1}{y_2 - \beta y_1} \right)^{\lambda} (y_2 - \beta y_1) = k. \tag{107}$$

Da nun y_1 und y_2 als Integrale von (78) auch Integrale von (95) sein müssen, so werden auch

$$y_2 - \beta y_1 = \eta_1 \text{ und } y_2 - \alpha y_1 = \eta_2$$

Integrale der linearen Differentialgleichung (95) sein müssen und nach (107) in der Beziehung stehen

$$\eta_2 = K.\eta_1^\sigma, \tag{108}$$

worin σ eine rationale Zahl und K eine algebraische Function von x bedeutet.* Setzt man aber diesen Werth von η_2 im (95) ein, so folgt, weil η_1 ebenfalls (95) genügen muss, also

$$\eta_2' = K'\eta_1^\sigma + \sigma K\eta_1^{\sigma-1}\eta_1',$$

$$\eta_2'' = K''\eta_1^\sigma + 2\sigma K'\eta_1^{\sigma-1}\eta_1' + \sigma(\sigma-1)K\eta_1^{\sigma-2}\eta_1'^2 - \sigma K\eta_1^{\sigma-1}(f_1(x)\eta_1' + f_2(x)\eta_1)$$

---

* Wenn die Lösungen α und β der Gleichung $Au^2 + (B+C)u + D = 0$ einander gleich sind, so nimmt die Partialbruchzerlegung die Form an

$$\frac{Au+B}{Au^2+(B+C)u+D} = \frac{1}{u-\alpha} + \frac{B+A\alpha}{A}\,\frac{1}{(u-\alpha)^2}$$

und somit

$$\log k - \log y_1 = \log(u-\alpha) - \frac{B+A\alpha}{A}\,\frac{1}{u-\alpha};$$

da aber u eine algebraische Function von x und y_1 sein soll, so muss $B + A\alpha = \frac{B-C}{2} = 0$ also $B = C$ sein, und somit $\frac{k}{y_1} = \frac{y_2 - \alpha y_1}{y_1}$ oder $y_2 - \alpha y_1 = k$ sein, d. h. es würde die Differentialgleichung zweiter Ordnung ein algebraisches Integral besitzen, das von Null verschieden ist, da $y_2 = \alpha y_1$ die Differentialgleichung erster Ordnung schon als eine homogene lineare definiren würde, für welche die Frage der algebraisch ausführbaren Quadraturen bereits oben erledigt worden. Da nun zwischen y_1 und y_2 die algebraische Beziehung (97) stattfinden sollte, so müssten vermöge der Annahme $y_2 - \alpha y_1 = k$ auch y_1 und y_2 selbst algebraische Functionen von x sein und die Frage nach algebraisch ausdrückbaren Quadraturen $\int y_1 dx$ wäre dann durch den bekannten Abel'schen Satz erledigt.

Wäre endlich $A = 0$, so ergäbe sich

$$\log k - \log y_1 = B\int\frac{du}{(B+C)u+D} = \frac{B}{B+C}\log\left(u + \frac{D}{B+C}\right)$$

oder

$$\frac{k}{y_1} = \left(u + \frac{D}{B+C}\right)^{\frac{B}{B+C}},$$

woraus der Annahme gemäss sich $\frac{B}{C}$ als rationale Zahl ergeben muss, und somit

$$\frac{k}{y_1} = (y_2 + hy_1)^{\lambda} y_1^{-\lambda},$$

worin λ eine rationale Zahl, h eine Constante bedeutet, oder endlich, wenn

$$y_2 + hy_1 = \eta_2,\ y_1 = \eta_1$$

gesetzt wird, worin η_1 und η_2 Integrale der homogenen linearen Differentialgleichung zweiter Ordnung sein werden,

$$\eta_2 = K\eta_1^\sigma,$$

worin σ eine rationale Zahl und K eine algebraische Function von x bedeutet; dies wäre aber wieder die oben gefundene Beziehung (108), so dass die weiteren Schlüsse dieselben wie oben bleiben.

ist, die Beziehung

$$\eta_1'^2 + 2\frac{K'}{K}\,\frac{\eta_1}{\sigma-1}\,\eta_1' = \frac{\eta_1^2}{K\sigma(\sigma-1)}\left[f_2(x)K(\sigma-1) - f_1(x)K' - K''\right]$$

oder

$$\eta_1' = L\eta_1, \tag{109}$$

worin L eine algebraische Function von x bedeutet, und somit nach (108)

$$\eta_2' = \left(\sigma L + \frac{K'}{K}\right)\eta_2; \tag{110}$$

es genügen somit die beiden particulären Integrale η_1 und η_2 zwei *linearen homogenen* Differentialgleichungen erster Ordnung, und wir finden zunächst,

dass, wenn die Quadratur eines Integrales einer homogenen linearen Differentialgleichung algebraisch durch dieses Integral ausdrückbar sein soll, die Differentialgleichung zwei Integrale besitzen muss, welche linearen homogenen Differentialgleichungen erster Ordnung Genüge leisten.

Da nun der Voraussetzung gemäss für das Integral y_1 der Differentialgleichung erster Ordnung (78) die Beziehung

$$\int y_1 dx = F(x, y_1) \tag{111}$$

stattfinden sollte, daraus aber nach oben bewiesenen Sätzen für das Integral y_2 derselben Differentialgleichung

$$\int y_2 dx = F(x, y_2) \tag{112}$$

folgt, so ergiebt sich

$$\int (y_2 - \beta y_1)\, dx = F(x, y_2) - \beta F(x, y_1) \tag{113}$$

oder vermöge der obigen Substitutionsgleichungen und der algebraischen Beziehung (108)

$$\int \eta_1 dx = \phi(x, \eta_1), \tag{114}$$

worin ϕ eine algebraische Function bedeutet. Da aber η_1 ein Integral der linearen homogenen Differentialgleichung erster Ordnung

$$\eta' = L\eta \tag{115}$$

ist, so folgt aus dem oben für lineare Differentialgleichungen erster Ordnung entwickelten Satze als nothwendige und hinreichende Bedingung für die algebraisch ausführbare Quadratur (114), dass

$$\int e^{\int L dx} dx = P_1 e^{\int L dx} \tag{116}$$

ist, worin P_1 eine rationale Function von x bedeutet, oder dass die Gleichung

$$t' + Lt = 1$$

ein rationales Integral P_1 besitze, und die Gleichung (116) ist zugleich die Reductionsformel der Quadratur. Dasselbe gilt für η_2, welches ein Integral der homogenen linearen Differentialgleichung erster Ordnung

$$\eta' = \left(\sigma L + \frac{K'}{K}\right)\eta \tag{117}$$

ist, und wir erhalten die Bedingung

$$\int e^{\int\left(\sigma L + \frac{K'}{K}\right)dx} = Q_1 e^{\int\left(\sigma L + \frac{K'}{K}\right)dx} \text{ oder } \int K e^{\sigma\int L dx} dx = Q_1 . K e^{\sigma \int L dx} \tag{118}$$

worin Q_1 rational aus x zusammengesetzt ist.

Damit aber ein Integral der homogenen linearen Differentialgleichung zweiter Ordnung

$$y'' + y' f_1(x) + f_2(x)\, y = 0 \tag{119}$$

der Differentialgleichung (115) Genüge leiste, ist offenbar nothwendig und hinreichend, dass, weil

$$\eta'' = L'\eta + L^2\eta, \quad \eta' = L\eta$$

ist,

$$L' + L^2 + f_1(x)\, L + f_2(x) = 0 \tag{120}$$

ist, oder dass die Differentialgleichung

$$u' + u^2 + f_1(x)\, u + f_2(x) = 0 \tag{121}$$

ein algebraisches Integral hat, und wir erhalten somit, wenn wir die gewonnenen Resultate zusammenfassen, den nachfolgenden Satz:

Die nothwendigen und hinreichenden Bedingungen dafür, dass eine lineare homogene Differentialgleichung zweiter Ordnung

$$y'' + f_1(x)\, y' + f_2(x)\, y = 0, \tag{α}$$

zwischen deren beiden Fundamentalintegralen eine algebraische Beziehung stattfindet, die Eigenschaft besitzt, dass die Quadratur über eines ihrer Integrale algebraisch durch eben dieses Integral ausdrückbar ist, sind die, dass die Differentialgleichung erster Ordnung

$$u' + u^2 + f_1(x)\, u + f_2(x) = 0 \tag{β}$$

ein algebraisches Integral L besitzt, und dass die lineare Differentialgleichung erster Ordnung

$$t' + Lt = 1 \tag{γ}$$

durch eine rationale Function P_1 *von* x *integrirbar ist; es lautet dann die Reductionsformel der Quadratur für das Integral*

$$y = e^{\int L dx}$$

der linearen Differentialgleichung zweiter Ordnung

$$\int e^{\int L dx} dx = P_1 e^{\int L dx}. \tag{δ}$$

Will man solche Fälle aufstellen, so braucht man nur für t irgend eine rationale Function zu wählen, aus (γ) L zu berechnen, diesen Werth für u in (β) einzusetzen und $f_1(x), f_2(x)$ so zu wählen, dass der Gleichung (β) Genüge geschieht.

Wenn jedoch zwischen den particulären Integralen der Differentialgleichung zweiter Ordnung keine algebraische Beziehung stattfindet, so genügte nach dem Obigen das Integral y_1 derselben, für welches die Reductionsformel

$$\int y_1 dx = F(x, y_1) \tag{122}$$

stattfinden sollte, einer linearen Differentialgleichung erster Ordnung

$$y' = F_1(x) y + F_2(x), \tag{123}$$

und umgekehrt, damit dies stattfindet, muss wegen

$$\begin{aligned} y'' &= F_1'(x)\, y + F_2'(x) + F_1(x)\, [F_1(x)\, y + F_2(x)] \\ &= [F_1'(x) + F_1^2(x)]\, y + F_2'(x) + F_1(x)\, F_2(x) \end{aligned}$$

nach (α)

$$[F_1'(x) + F_1^2(x) + f_1(x)\, F_1(x) + f_2(x)]\, y + F_2'(x) + F_1(x)\, F_2(x) + f_1(x)\, F_2(x) = 0$$

sein oder, da y keine algebraische Function von x sein sollte,

$$\begin{aligned} F_1'(x) + F_1^2(x) + f_1(x)\, F_1(x) + f_2(x) &= 0, \\ F_2'(x) + F_1(x)\, F_2(x) + f_1(x)\, F_2(x) \quad &= 0, \end{aligned}$$

und man erhält somit zunächst als nothwendige und hinreichende Bedingung dafür, dass ein Integral der linearen homogenen Differentialgleichung zweiter Ordnung (α) ein Integral mit einer linearen Differentialgleichung erster Ordnung gemein hat, die, dass die beiden Differentialgleichungen

$$t' + t^2 + f_1(x)\, t + f_2(x) = 0$$

und

$$u' + tu + f_1(x)\, u = 0$$

algebraische Integrale besitzen, wobei das t der zweiten Differentialgleichung das algebraische Integral der ersten bedeutet. Dafür aber dass ein Integral der Gleichung (123) eine algebraisch ausführbare Quadratur besitzt, sind oben die nothwendigen und hinreichenden Bedingungen aufgestellt worden und wir erhalten somit den folgenden Satz:

Die nothwendigen und hinreichenden Bedingungen dafür, dass eine Differentialgleichung

$$y'' + y'f_1(x) + yf_2(x) = 0,$$

deren Fundamentalintegrale nicht in algebraischem Zusammenhange stehen, eine für ein Integral derselben algebraisch ausführbare Quadratur besitzt, sind zunächst die, dass die Differentialgleichungen

$$t' + t^2 + f_1(x)t + f_2(x) = 0,$$
$$u' + tu + f_1(x)u = 0$$

algebraische Integrale $F_1(x)$ und $F_2(x)$ besitzen, dass ferner der Differentialgleichung erster Ordnung

$$v' + vF_1(x) = 1$$

ein rationales Integral P_1 angehört und die Quadratur

$$\int P_1F_2(x)\,dx = -P_2$$

eine rationale Function von x ist; sind diese Bedingungen erfüllt, so lautet die Reductionsformel der Quadratur

$$\int y dx = P_1y + P_2.$$

Es ist endlich aus dem Obigen ersichtlich, dass die algebraischen Functionen $F_1(x)$ und $F_2(x)$ rationale Functionen von x sein mussten.

Nachdem wir an einzelnen Fällen linearer Differentialgleichungen gezeigt haben, wie die oben von algebraisch ausführbaren Quadraturen von Integralen algebraischer Differentialgleichungen bewiesenen Sätze zur Ermittlung der nothwendigen und hinreichenden Bedingungen für diese Ausdrückbarkeit zu benutzen sind, wenden wir uns wieder zu den allgemeinen Untersuchungen zurück und werden *die Quadratur*

$$\int y_1 dx = A \log z_1, \tag{124}$$

wenn y_1 *ein Integral der Differentialgleichung* (1) *oder* (2) *ist, eine logarithmisch ausführbare nennen, wenn* z_1 *durch die Gleichung* (4) *oder* (5) *definirt ist und* A *eine Constante bedeutet.*

Aus dem durch die Gleichung (9) dargestellten Ausdrucke für z_1' folgt durch Zusammenstellung mit (124) oder

$$z_1' = z_1 y_1 \tag{125}$$

die Beziehung

$$P_0 z_1^{\mu-1} + P_1 z_1^{\mu-2} + \dots\dots + (P_{\mu-2} - y_1) z_1 + P_{\mu-1} = 0, \tag{126}$$

welche, weil die Gleichung (5) als eine mit Adjungirung von $x, y_1, y_2, \dots\dots$ und deren Ableitungen bis zur m^{ten} Ordnung hin algebraisch irreductible vorausgesetzt war,

$$P_0 = 0 \quad P_1 = 0 \dots\dots P_{\mu-2} - y_1 = 0 \quad P_{\mu-1} = 0$$

liefert, so dass sich für jede andere Lösung z_α der Gleichung (5)

$$z_\alpha' = z_\alpha y_1 \tag{127}$$

ergiebt und daher

$$\frac{z_2'}{z_\alpha} = \frac{z_1'}{z_1} \text{ oder } z_\alpha = c_\alpha z_1 \tag{128}$$

folgt. Da jedoch das Zusammenbestehen der aus (5) und (128) sich ergebenden Gleichungen

$$z_1^\mu + f_1 z_1^{\mu-1} + f_2 z_1^{\mu-2} + \dots\dots + f_{\mu-1} z_1 + f_\mu = 0, \tag{129}$$

$$c_\alpha^\mu z_1^\mu + c_\alpha^{\mu-1} f_1 z_1^{\mu-1} + c_\alpha^{\mu-2} f_2 z_1^{\mu-2} + \dots\dots + c_\alpha f_{\mu-1} z_1 + f_\mu = 0 \tag{130}$$

wegen der Irreductibilität der Gleichung (5) aus

$$c_\alpha^{\mu-1}(c_\alpha - 1) f_1 z_1^{\mu-1} + c_\alpha^{\mu-2}(c_\alpha^2 - 1) f_2 z_1^{\mu-2} + \dots\dots + c_\alpha (c_\alpha^{\mu-1} - 1) f_{\mu-1} z_1 + (c_\alpha^\mu - 1) f_\mu = 0$$

die nothwendigen Bedingungen ergiebt

$$(c_\alpha - 1) f_1 = 0,\ (c_\alpha^2 - 1) f_2 = 0, \dots\dots (c_\alpha^{\mu-1} - 1) f_{\mu-1} = 0,\ (c_\alpha^\mu - 1) f_\mu = 0,$$

so folgt, da nicht alle $f_1 . f_2, \dots\dots f_\mu = 0$ sein können, dass c_α eine δ^{te} Einheitswurzel ist, also

$$c_\alpha^\delta = 1$$

sein wird, worin $1 < \delta \leqq \mu$,* und man sieht zugleich, dass, weil f_μ nicht verschwinden darf, da sonst die irreductible Gleichung (5) die Lösung Null hätte, $c_\alpha^\mu = 1$ also μ ein ganzes Multiplum von δ sein muss, welches wir $= \lambda . \delta$ setzen wollen, so dass alle f, deren Index nicht ein Multiplum von δ ist, verschwinden müssen, und die Gleichung (129) die Form annimmt

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* Die Annahme $\delta = 1$ würde $c_\alpha = 1$ also $z_\alpha = z_1$ liefern, was für irreductible Gleichungen unmöglich ist.

$$z_1^{\lambda\delta} + f_\delta z_1^{(\lambda-1)\delta} + f_{2\delta} z_1^{(\lambda-2)\delta} + \dots + f_{\lambda\delta} = 0; \tag{131}$$

setzt man nun

$$z_1^\delta = t_1, \tag{132}$$

so folgt aus (124)

$$\int y_1 dx = \frac{A}{\delta} \log t_1, \tag{133}$$

worin t_1 eine Lösung der Gleichung

$$t^\lambda + f_\delta t^{\lambda-1} + f_2 t^{\lambda-2} + \dots + f_{\lambda\delta} = 0, \tag{134}$$

deren Grad $\lambda < \mu$ ist. Auf die Zusammenstellung der Gleichungen (133) und (134) können wir genau dieselben Schlüsse anwenden, die wir für (124) und (5) gemacht haben, und da der Grad der den Logarithmanden definirenden Gleichung immer verkleinert wird, derselben also auch auf die Einheit gebracht werden kann, der Logarithmand dann also rational in den in Betracht kommenden Grössen ausdrückbar ist, so erhalten wir den folgenden Satz:

Wenn die Quadratur

$$\int y_1 dx,$$

worin y_1 ein Integral einer algebraischen Differentialgleichung m^{ter} Ordnung bedeutet, logarithmisch ausführbar ist, so lässt sich der Werth derselben stets als ein mit einer multiplicatorischen Constanten versehener Logarithmus darstellen, dessen Logarithmand eine RATIONALE *Function von x, particulären Integralen der Differentialgleichung und deren Ableitungen bis einschliesslich zur m^{ten} Ordnung hin ist, die sich wiederum in eine in den m^{ten} Ableitungen der Integrale ganze Function $n - 1^{\text{ten}}$ Grades für jede derselben umsetzen lässt, deren Coefficienten rationale Functionen von x, den Integralen der Differentialgleichung und deren Ableitungen bis zur $m - 1^{\text{ten}}$ Ordnung hin sind.*

Sei nun

$$\int y_1 dx = A \log F(x, y_1, y_1', \dots y_1^{(m)}, y_2, y_2', \dots y_2^{(m)}, \dots), \tag{135}$$

worin F den eben angegebenen Charakter hat, so liefert die Differentiation dieser Gleichung

$$\begin{aligned} y_1 F(x, y_1, y_1', \dots y_1^{(m)}, y_2, y_2', \dots y_2^{(m)}, \dots) \\ = A\Big(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y_1} y_1' + \dots + \frac{\partial F}{\partial y_1^{(m)}} y_1^{(m+1)} + \frac{\partial F}{\partial y_2} y_2' \\ + \dots + \frac{\partial F}{\partial y_2^{(m)}} y_2^{(m+1)} + \dots\Big); \end{aligned} \tag{136}$$

da nun $y_1^{(m+1)}$, $y_2^{(m+1)}$, ... rational durch die niedrigeren Ableitungen vermöge der Gleichung (1) ausdrückbar sind, so wird die Gleichung (136) nach dem oben bewiesenen allgemeinen Satze bestehen bleiben, wenn statt y_1 ein jedes nicht singuläre Integral η_1 der Differentialgleichung (35), das auch ein Integral von (1) ist, gesetzt wird, während $y_2, y_3, \ldots$ durch passende Integrale $\eta_2, \eta_3 \ldots$ der Differentialgleichung (1) zu ersetzen sind, und es folgt somit

$$\eta_1 F(x, \eta_1, \eta_1', \ldots \eta_2, \eta_2', \ldots)$$
$$= A\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial \eta_1}\eta_1' + \ldots + \frac{\partial F}{\partial \eta_1^{(m)}}\eta_1^{(m+1)} + \frac{\partial F}{\partial \eta_2}\eta_2' + \ldots + \frac{\partial F}{\partial \eta_2^{(m)}}\eta_2^{(m+1)}, \ldots\right)$$

oder
$$\int \eta_1 dx = A \log F(x, \eta_1, \eta_1', \ldots \eta_1^{(m)}, \eta_2, \eta_2', \ldots \eta_2^{(m)}, \ldots)$$

und daraus der folgende Satz:

Wenn die Quadratur eines Integrales einer algebraischen Differentialgleichung m^{ter} Ordnung logarithmisch ausführbar ist, so ist es auch die Quadratur eines jeden anderen Integrales eben dieser Differentialgleichung, welches ein beliebiges nicht singuläres Integral der in Bezug auf den höchsten Differentialquotienten algebraisch irreductibeln Differentialgleichung niedrigster Ordnung ist, welcher jenes Integral Genüge leistet, und zwar in Form des mit derselben Constanten multiplicirten Logarithmus derselben ganzen Function der m^{ten} Ableitungen von Integralen der gegebenen Differentialgleichung m^{ter} Ordnung mit Coefficienten, welche rationale Functionen der unabhängigen Variabeln, der vorkommenden Integrale und deren Ableitungen bis zur $m-1^{ten}$ Ordnung hin sind, wobei diese Integrale passend zu wählen sind.

Das Integral η_1 darf wiederum *jedes* nicht singuläre Integral der Differentialgleichung (1) bedeuten, wenn diese letztere in Bezug auf die höchste Ableitung algebraisch irreductibel ist und die Basis der Quadratur y_1 nicht schon einer Differentialgleichung von niederer Ordnung als der m^{ten} Genüge leistet.

Ebenso ist wie oben für algebraisch ausführbare Quadraturen ersichtlich, dass, weil aus
$$\int y_1 dx = A \log F(x, y_1, y_1', \ldots y_1^{(m-a)})$$

durch Differentiation
$$y_1 F(x, y_1, y_1', \ldots y_1^{(m-a)}) = A\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y_1}y_1' + \ldots + \frac{\partial F}{\partial y_1^{(m-a)}}y_1^{(m-a+1)}\right)$$
folgt,

wenn y_1 das Integral einer Differentialgleichung m^{ter} Ordnung ist, welches nicht schon einer Differentialgleichung niederer Ordnung Genüge leistet, eine logarithmisch

ausführbare Quadratur, welche nur das Basisintegral und dessen Ableitungen enthalten soll, nothwendig die m — 1ᵗᵉ Ableitung der Basis einschliessen muss,
und

soll die Quadratur irgend einer transcendenten Function y_1 sich logarithmisch durch eben diese und deren Ableitungen ausdrücken lassen, so muss diese das Integral einer algebraischen Differentialgleichung sein, und soll der Logarithmand nur die Transcendente enthalten, so wird diese durch eine Differentialgleichung erster Ordnung definirt.

Wir wollen, um eine Anwendung dieser Sätze zu machen, untersuchen, wann die Quadratur eines Integrales y_1 der linearen Differentialgleichung erster Ordnung

$$y' = yf_1(x) + f_2(x) \tag{137}$$

logarithmisch ausführbar ist, also

$$\int y_1 dx = A \log F(x, y_1, y_1'), \tag{138}$$

worin F eine rationale Function bedeutet oder nach (137)

$$\int y_1 dx = A \log \omega(x, y_1), \tag{139}$$

worin ω ebenfalls rational ist. Da nach dem Obigen jedes Integral von (137) in (139) eingesetzt werden darf, also

$$\int (y_1 + ce^{\int f_1(x)dx})\, dx = A \log \omega(x, y_1 + ce^{\int f_1(x)dx}) \tag{140}$$

wird, so folgt aus (139) und (140)

$$A \log \omega(x, y_1 + ce^{\int f_1(x)dx}) - A \log \omega(x, y_1) = c\int e^{\int f_1(x)dx} dx \tag{141}$$

für willkührliche Werthe der Integrationsconstanten c. Differentiirt man diese Gleichung nach c, so ergiebt sich

$$\frac{A d \log \omega(x, y_1 + ce^{\int f_1(x)dx})}{d(y_1 + ce^{\int f_1(x)dx})} = e^{-\int f_1(x)dx} \int e^{\int f_1(x)dx} dx \,. \tag{142}$$

und somit die linke Seite von c also auch vom ganzen Argumente $y_1 + ce^{\int f_1(x)dx}$ unabhängig und daher

$$\log \omega(x, y_1) = P_1 y_1 + P_2 \tag{143}$$

oder

$$\omega(x, y_1) = e^{P_2} . e^{P_1 y_1}, \tag{144}$$

worin P_1 und P_2 nur von x abhängen; da aber $\omega(x, y_1)$ eine rationale Function von x und y_1 sein sollte, so muss $P_1 = 0$ sein, was nicht möglich ist, da dann $\omega(x, y_1)$ von y_1 unabhängig wäre und die Beziehung

$$\int y_1 dx = A \log \Omega(x)$$

y_1 als rationale Function von x liefern würde, was ausgeschlossen war; wir finden daher

dass die Quadratur eines transcendenten Integrales einer linearen Differentialgleichung erster Ordnung nie logarithmisch ausführbar sein kann.

Ebenso kann man vermöge der obigen Untersuchungen leicht die Frage erörtern, ob die Quadratur eines Integrales y_1 einer linearen homogenen Differentialgleichung zweiter Ordnung

$$y'' + y'f_1(x) + yf_2(x) = 0 \tag{145}$$

so logarithmisch ausführbar ist, dass der Logarithmand eine algebraische Function von x und y_1 ist, also

$$\int y_1 dx = A \log F(x, y_1) \tag{146}$$

wird. Da nämlich aus (146) durch Differentiation

$$y_1 F(x, y_1) = A\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y_1} y_1'\right) \tag{147}$$

sich ergiebt, also y_1 die Lösung einer algebraischen Differentialgleichung erster Ordnung sein wird, so muss nach den oben gemachten Durchführungen entweder y_1 selbst einer linearen Differentialgleichung erster Ordnung von der Form

$$y' = yF_1(x) + F_2(x) \tag{148}$$

genügen, oder es wird, wenn zwischen y_1 und einem anderen Integral y_2 der Differentialgleichung erster Ordnung (147) eine algebraische Beziehung stattfindet, ein Integral

$$y_2 - \beta y_1 = \eta_1 \tag{149}$$

der Differentialgleichung (145) der linearen homogenen Differentialgleichung erster Ordnung

$$\eta' = L\eta \tag{150}$$

genügen. Im ersteren Falle ist nach der eben durchgeführten Untersuchung die logarithmische Ausführbarkeit der Quadratur unmöglich, im zweiten Falle werden die am Anfange der Arbeit bewiesenen Sätze die Beziehungen

$$\int y_1 dx = A \log F(x, y_1) \qquad \int y_2 dx = A \log F(x, y_2) \tag{151}$$

liefern, woraus

$$\int (y_2 - \beta y_1)\, dx = A\,\{\log F(x, y_2) - \beta \log F(x, y_1)\} \tag{152}$$

folgt; da aber vermöge der oben entwickelten Beziehung

$$y_2 - \beta y_1 = \eta_1,\; y_2 - \alpha y_1 = \eta_2,\; \eta_2 = K\eta_1^\sigma \tag{153}$$

sich y_2 und y_1 als algebraische Functionen von η_1 ergeben, so geht (152) in

$$\int \eta_1 dx = A\,\{\log \omega_0(x, \eta_1) - \beta \log \omega_1(x, \eta_1)\} \tag{154}$$

über. Da aber die Differentiation dieser Gleichung mit Benutzung von (150)

$$\eta_1 = A\left\{\frac{\partial \omega_0(x, \eta_1)}{\partial x} + \frac{\partial \omega_0(x, \eta_1)}{\partial \eta_1} L\eta_1\right\} : \omega_0(x, \eta_1) - A\beta\left\{\frac{\partial \omega_1(x, \eta_1)}{\partial x} + \frac{\partial \omega\eta_1(x, {}_1)}{\partial \eta_1} L\eta_1\right\}$$

liefert, und diese Beziehung, da η_1 nicht eine algebraische Function von x sein sollte, eine identische sein muss, so wird (154) für jedes Integral der Differentialgleichung (150) bestehen bleiben, und somit für willkührliche Werthe der Integrationsconstanten c

$$c\int \eta_1 dx = A\,\{\log \omega_0(x, c\eta_1) - \beta \log \omega_1(x, c\eta_1)\} \tag{155}$$

und aus (154) und (155)

$$\log \omega_0(x, c\eta_1) - \beta \log \omega_1(x, c\eta_1) = c\,\{\log \omega_0(x, \eta_1) - \beta \log \omega_1(x, \eta_1)\}, \tag{156}$$

welche Gleichung eine in x, η_1 und c identische sein muss. Differentiirt man nun (156) nach η_1 und c, so ergiebt sich

$$\frac{d \log \omega_0(x, c\eta_1)}{dc\eta_1} - \beta\,\frac{d \log \omega_1(x, c\eta_1)}{dc\eta_1} = \frac{d \log \omega_0(x, \eta_1)}{d\eta_1} - \beta\,\frac{d \log \omega_1(x, \eta_1)}{d\eta_1}$$

und

$$\frac{d \log \omega_0(x, c\eta_1)}{dc\eta_1} - \beta\,\frac{d \log \omega_1(x, c\eta_1)}{dc\eta_1} = \frac{1}{\eta_1}\,(\log \omega_0(x, \eta_1) - \beta \log \omega_1(x, \eta_1)),$$

woraus
$$\frac{d(\log \omega_0(x, \eta_1) - \beta \log \omega_1(x, \eta_1))}{\log \omega_0(x, \eta_1) - \beta \log \omega_1(x, \eta_1)} = \frac{d\eta_1}{\eta_1}$$

und daher

$$\log \omega_0(x, \eta_1) - \beta \log \omega_1(x, \eta_1) = \phi(x) . \eta_1 \tag{157}$$

oder nach (154)

$$\int \eta_1 dx = A\phi(x) . \eta_1, \tag{158}$$

worin $\phi(x)$ noch eine logarithmische Function von x wäre, und der Werth von $\phi(x)$ ergiebt sich aus (157), indem man $\eta = 1$ setzt und

$$\phi(x) = \log \omega_0(x, 1) - \beta \log \omega_1(x, 1) \tag{159}$$

also

$$\int \eta_1 dx = A\left(\log \omega_0(x, 1) - \beta \log \omega_1(x, 1)\right) \eta_1 \tag{160}$$

erhält. Durch Differentiation dieser Gleichung würde aber mit Berücksichtigung der Gleichung (150)

$$1 = AL\left(\log \omega_0(x, 1) - \beta \log \omega_1(x, 1)\right) + A\left(\frac{\omega_0'(x, 1)}{\omega_0(x, 1)} - \beta \frac{\omega_1'(x, 1)}{\omega_1(x, 1)}\right)$$

also

$$A\left(\log \omega_0(x, 1) - \beta \log \omega_1(x, 1)\right) = \Omega(x) \tag{161}$$

sich ergeben, worin $\Omega(x)$ eine algebraische Function bedeutet, und somit (160) in

$$\int \eta_1 dx = \Omega(x) \eta_1 \tag{162}$$

übergehen. Da nun vermöge

$$y_2 - \beta y_1 = \eta_1 \quad y_2 - \alpha y_1 = \eta_2$$

sich

$$y_1 = \frac{1}{\alpha - \beta}(\eta_1 - \eta_2)$$

ergiebt, und somit aus (151)

$$\int (\eta_1 - \eta_2) dx = A(\alpha - \beta) \log F\left(x, \frac{\eta_1 - \eta_2}{\alpha - \beta}\right)$$

folgt oder nach (108) und (162)

$$\int \eta_2 dx = -A(\alpha - \beta) \log F\left(x, K^{-\frac{1}{\sigma}} \eta_2^{\frac{1}{\sigma}}\right) + \Omega(x) K^{-\frac{1}{\sigma}} \eta_2^{\frac{1}{\sigma}}, \tag{163}$$

worin σ eine rationale Zahl und K eine algebraische Function ist, ferner η_2 die Lösung der linearen homogenen Differentialgleichung erster Ordnung (110) ist, so wäre zu untersuchen, wann die Quadratur des Integrales einer homogenen linearen Differentialgleichung erster Ordnung

$$z' = \omega(x) z \tag{164}$$

algebraisch-logarithmisch in der Form

$$\int z_1 dx = B \log \phi(x, z_1) + \psi(x, z_1) \tag{165}$$

ausdrückbar ist, wenn ϕ und ψ algebraische Functionen bedeuten. Beachtet man aber, dass nach den oben angegebenen Sätzen die Beziehung (165) erhalten bleiben muss, wenn statt z_1 das allgemeine Integral cz_1 von (164) gesetzt wird, so folgt

$$c\int z_1 dx = B \log \phi(x, cz_1) + \psi(x, cz_1) \tag{166}$$

und aus (165) und (166)

$$B \log \phi(x, cz_1) + \psi(x, cz_1) = cB \log \phi(x, z_1) + c\psi(x, z_1). \tag{167}$$

Da nun diese Beziehung für jedes c gelten muss, so wird also auch nach c differentiirt werden dürfen, und man erhält

$$B\frac{\frac{\partial \phi(x, cz_1)}{\partial cz_1}}{\phi(x, cz_1)} z_1 + \frac{\partial \psi(x, cz_1)}{\partial cz_1} z_1 = B \log \phi(x, z_1) + \psi(x, z_1), \tag{168}$$

wonach die rechte Seite also auch nach (165) der Werth der Quadratur selbst eine algebraische Function von x und z_1 wäre, wir finden somit,

dass die Quadratur eines Integrales einer linearen homogenen Differentialgleichung erster Ordnung nie algebraisch-logarithmisch ausführbar sein kann,

und dass somit also auch oben (163) in

$$\int \eta_2 dx = \chi(x, \eta_2) \tag{169}$$

übergeht, wonach sich aus (162) und (169)

$$\int y_1 dx = \frac{1}{\alpha - \beta}\int (\eta_1 - \eta_2)\, dx = \frac{1}{\alpha - \beta}(\Omega(x)\eta_1 - \chi(x, \eta_2)) = P(x, y_1) \tag{170}$$

ergiebt, worin $P(x, y_1)$ eine algebraische Function von y_1 darstellt, also der Annahme (151) widerspricht. Wir erhalten somit den folgenden Satz:

Die Quadratur eines transcendenten Integrales einer homogenen linearen Differentialgleichung zweiter Ordnung ist nie logarithmisch ausdrückbar.

Nachdem ich, um die Verallgemeinerung der von Abel für Integrale algebraischer Functionen aufgestellten Sätze für Quadraturen von Integralen algebraischer Differentialgleichungen zu entwickeln, die algebraisch und logarithmisch ausdrückbaren Quadraturen untersucht und die Anwendung der gefundenen Sätze auf bestimmte Probleme gezeigt habe, werde ich jetzt diese Frage ganz allgemein angreifen.

Ich nenne eine Quadratur eines Integrales einer algebraischen Differentialgleichung im Abel'schen Sinne ausführbar, wenn

$$\int y_1 dx = F\left(x, \log v_1, \log v_2, \ldots . \log v_\rho, \int^{s_1} V_1 ds, \int^{s_2} V_2 ds, \ldots . \int^{s_\sigma} V_\sigma ds\right), \quad (171)$$

worin $V_1, V_2, \ldots . V_\sigma$ *algebraische Functionen von* s, *und* $v_1, v_2, \ldots . v_\rho, s_1, s_2, \ldots . s_\sigma$ *algebraische Functionen von* x *und den Integralen* $y_1, y_2, \ldots .$ *der algebraischen Differentialgleichung* (1) *oder* (2) *und deren Ableitungen bis zur* m^{ten} *Ordnung hin sind, die auch algebraisch in* F *eintreten können,*

diese letzteren algebraischen Functionen wollen wir uns durch die Gleichungen definirt denken

$$v^{\varkappa_\alpha} + f_{1\alpha}(x, y_1, y_1', \ldots . y_1^{(m)}, y_2, y_2', \ldots . y_2^{(m)}, \ldots .)\, v^{\varkappa_\alpha - 1}$$
$$+ \ldots . + f_{\varkappa_\alpha \alpha}(x, y_1, y_1', \ldots . y_2, y_2', \ldots .) = 0 \quad (172)$$

und

$$s^{\lambda_\beta} + \phi_{1\beta}(x, y_1, y_1', \ldots . y_1^{(m)}, y_2, y_2', \ldots . y_2^{(m)}, \ldots .)\, v^{\lambda_\beta - 1}$$
$$+ \ldots . + \phi_{\lambda_\beta \beta}(x, y_1, y_1', \ldots . y_2, y_2', \ldots .) = 0, \quad (173)$$

die mit Adjungirung der eingeschlossenen Grössen als algebraisch irreductibel vorausgesetzt werden mögen, und es kann endlich von den in (171) enthaltenen logarithmischen und Abel'schen Transcendenten angenommen werden, dass mit Adjungirung von $y_1, y_2, \ldots .$ und deren Ableitungen kein algebraischer Zusammenhang zwischen ihnen stattfindet, weil im entgegengesetzten Falle mit Hülfe desselben die rechte Seite von (171) vereinfacht und in dieser neuen Form der Untersuchung zu Grunde gelegt werden könnte.

Es wird die Frage aufgeworfen, unter welchen Bedingungen und in welcher Form die Quadratur einer algebraischen Differentialgleichung im Abel'schen Sinne ausführbar ist?

Die Differentiation der Gleichung (171) liefert

$$y_1 = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial \log v_1} \frac{v_1'}{v_1} + \ldots . + \frac{\partial F}{\partial \int^{s_1} V_1 ds} V_1(s_1) s_1' + \ldots . \quad (174)$$

worin sich $v_\alpha', \ldots . s_\beta', \ldots .$ vermöge der Gleichungen (172) und (173) als ganze Functionen resp. des $\varkappa_\alpha - 1$, $\lambda_\beta - 1^{\text{ten}}$ Grades von v_α, s_β ausdrücken lassen mit Coefficienten, die rational aus $y_1, y_1', \ldots . y_1^{(m)}, y_2, y_2', \ldots .$ zusammengesetzt sind, und da (174) gegen die Voraussetzung einen algebraischen Zusammenhang zwischen den logarithmischen und Abel'schen Transcendenten liefern würde also in all' diesen Grössen identisch sein muss, so wird dieselbe auch befriedigt sein, wenn man

$$\log v_\alpha, \int^{s_\beta} V_\beta ds \text{ durch } \log v_\alpha + \mu_\alpha, \int^{s_\beta} V_\beta ds + v_\beta$$

ersetzt, und man sicht sogleich, dass man von (174) durch Integration unmittelbar zu der Gleichung

$$\int y_1 dx = F\left(x_1 \log v_1 + \mu_1, \ldots . \log v_\rho + \mu_\rho, \int^{s_1} V_1 ds + \nu_1, \ldots . \int^{s_\sigma} V_\sigma ds + \nu_\sigma\right) \quad (175)$$

übergehen kann, so dass, da die linken Seiten von (171) und (175) sich nur durch eine additive Constante unterscheiden können, für willkührliche Werthe von $\mu_1, \ldots . \mu_\rho, \nu_1, \ldots . \nu_\sigma$

$$F\left(x, \log v_1 + \mu_1, \ldots . \log v_\rho + \mu_\rho, \int^{s_1} V_1 ds + \nu_1, \ldots . \int^{s_\sigma} V_\sigma ds + \nu_\sigma\right)$$
$$= F\left(x, \log v_1, \ldots . \log v_\rho, \int^{s_1} V_1 ds, \ldots . \int^{s_\sigma} V_\sigma ds\right) + M \quad (176)$$

sein wird, worin M eine von den μ und ν abhängige Constante bedeutet.

Da nun diese Gleichung wieder gegen die Voraussetzung eine in den logarithmischen und Abel'schen Transcendenten algebraische Beziehung liefern würde, so muss dieselbe in all' diesen Grössen identisch sein und somit

$$\frac{\partial F\left(x, \log v_1 + \mu_1, \ldots . \log v_\rho + \mu_\rho, \int^{s_1} V_1 ds + \nu_1, \ldots . \int^{s_\sigma} V_\sigma ds + \nu_\sigma\right)}{\partial(\log v_\alpha + \mu_\alpha)}$$
$$= \frac{\partial F\left(x, \log v_1, \ldots . \log v_\rho, \int^{s_1} V_1 ds, \ldots . \int^{s_\sigma} V_\sigma ds\right)}{\partial \log v_\alpha}$$

und wegen der Willkührlichkeit der μ und ν aus demselben Grunde

$$F\left(x, \log v_1, \ldots . \log v_\rho, \int^{s_1} V_1 ds, \ldots . \int^{s_\sigma} V_\sigma ds\right) = P \log v_\alpha + Q,$$

worin P und Q im Allgemeinen noch von den anderen Transcendenten abhängen; da aber nach (176) für

$$\mu_1 = \ldots . = \mu_{\alpha-1} = \mu_{\alpha+1} = \ldots . = \mu_\rho = \nu_1 = \ldots . = \nu_\sigma = 0$$
$$P(\log v_\alpha + \mu_\alpha) + Q = P \log v_\alpha + Q + M$$

oder
$$P\mu_a = M$$

folgt und M eine Constante ist, so wird P ebenfalls constant sein müssen, und da dieselben Schlüsse für alle in F enthaltenen Transcendenten gelten, so ergiebt sich der folgende Satz:

Ist die Quadratur eines Integrales y_1 einer algebraischen Differentialgleichung m^{ter} Ordnung im Abel'schen Sinne ausführbar, so ist dieselbe eine mit constanten Coefficienten versehene lineare Function logarithmischer und Abel'scher Transcendenten von der Form

$$\int y_1 dx = u + A_1 \log v_1 + \dots + A_\rho \log v_\rho + B_1 \int^{s_1} V_1 ds + \dots + B_\sigma \int^{s_\sigma} V_\sigma ds, \tag{177}$$

worin $V_1, V_2, \dots V_\sigma$ algebraische Functionen von s, $A_1, \dots A_\rho$, $B_1, \dots B_\sigma$ Constanten, und $u, v_1, \dots v_\rho$, $s_1, \dots s_\sigma$ algebraische Functionen von x und den Integralen $y_1, y_2, \dots$ der algebraischen Differentialgleichung und deren Ableitungen bis zur m^{ten} Ordnung hin sind.

Bevor wir nun zur weiteren Reduction der Form (177) übergehen, wollen wir einen im Princip schon von Abel aufgestellten Satz ein wenig verallgemeinern, den Beweis desselben aber speciell für unser Problem durchführen.

Seien die algebraischen Functionen

$$u,\ v_a,\ s_a \text{ von } x,\ y_1, \dots y_1^{(m)},\ y_2, \dots y_2^{(m)}, \dots$$

durch die mit Adjungirung eben dieser transcendenten Grössen irreductibeln Gleichungen definirt

$$\left.\begin{aligned} u^\lambda + f_1 u^{\lambda-1} + \dots + f_\lambda &= 0, \\ v^{\mu_a} + \phi_{1a} v^{\mu_a - 1} + \dots + \phi_{\mu_a a} &= 0, \\ s^{\kappa_a} + \psi_{1a} s^{\kappa_a - 1} + \dots + \psi_{\kappa_a a} &= 0, \end{aligned}\right\} \tag{178}$$

in denen f, ϕ, ψ rationale Functionen der angegebenen Grössen bedeuten, ferner, wenn wir die in den von s algebraisch abhängenden Functionen $V_1, \dots V_\sigma$ vorkommenden algebraischen Irrationalitäten mit $\omega_1(s), \dots \omega_\sigma(s)$ bezeichnen, so dass V_a eine rationale Function von s und $\omega_a(s)$ darstellt, die $\omega_a(s_a)$ definirende, ebenfalls mit Adjungirung der angegebenen Grössen irreductible Gleichung

$$\omega^{\rho_a}(s) + \chi_{1a} \omega^{\rho_a - 1}(s) + \dots + \chi_{\rho_a a} = 0, \tag{179}$$

worin die χ wiederum rationale Functionen von x, $y_1, \dots y_1^{(m)}$, $y_2, \dots y_2^{(m)}, \dots$

bedeuten, so bilde man mit constanten Coefficienten den in jenen Grössen linearen Ausdruck

$$t_1 = au + b_1 v_1 + \dots + b_\rho v_\rho + c_1 s_1 + \dots + c_\sigma s_\sigma + d_1 \omega_1(s_1) + \dots + d_\sigma \omega_\sigma(s_\sigma). \quad (180)$$

Wir wollen uns zunächst die mit Adjungirung der Grössen $x, y_1, \dots y_1^{(m)}, y_2, \dots y_2^{(m)}$ algebraische irreductible Gleichung bilden, von welcher t_1 eine Lösung ist und haben zu dem Zwecke offenbar nur zwischen den für die verschiedenen Indices sich ergebenden Gleichungen (178), (179) und der Gleichung (180) die Grössen $u, v_\alpha, s_\alpha, \omega_\alpha(s_\alpha)$ zu eliminiren, und aus dieser Gleichung den mit Adjungirung von $x, y_1, \dots y_1^{(m)}, y_2, \dots y_2^{(m)}, \dots$ irreductibeln Theil abzusondern, welcher t_1 als Lösung enthält und durch

$$t^p + \omega_1(x, y_1, \dots y_1^{(m)}, y_2 \dots y_2^{(m)}, \dots) t^{p-1} + \dots$$
$$+ \omega_p(x, y_1, \dots y_1^{(m)}, y_2, \dots y_2^{(m)}, \dots) = 0 \quad (181)$$

dargestellt sein mag. Dass diese p Lösungen der Gleichung (181), von welcher eine t_1 ist, aus (179) erhalten werden, in dem man auf der rechten Seite für die $u, v, s, \omega(s)$ eine andere Combination der Lösungen der Gleichungen (178) und (179) setzt, geht daraus hervor, dass die Lösungen der irreductibeln Gleichung (181) dadurch entstehen, dass man x solche geschlossene, nicht durch Verzweigungspunkte gehende Umläufe machen lässt, dass die Coefficienten ihre Werthe wieder erlangen, und es ist klar, dass durch solche Umläufe Lösungen der Gleichungen (178) und (179) immer nur in Lösungen derselben Gleichung übergehen können, so dass den p Werthen von t gewisse Combinationen der Lösungen derjenigen Gleichung entsprechen werden, welche alle Lösungen der Gleichungen (177) und (178) zu Lösungen hat und die man erhält, wenn man die linken Seiten aller für dieselbe unbestimmte Variable genommenen Gleichungen (178) und (179) mit einander multiplicirt und welche die Form haben mag

$$\xi^M + F_1(x, y_1, \dots y_1^{(m)}, y_2, \dots y_2^{(m)}, \dots) \xi^{M-1} + \dots$$
$$+ F_M(x, y_1, \dots y_1^{(m)}, y_2, \dots y_2^{(m)}, \dots) = 0.$$

Stellen wir nun für irgend welche andere Constanten den linearen Ausdruck

$$T_1 = au + \beta_1 v_1 + \dots + \beta_\rho v_\rho + \gamma_1 s_1 + \dots + \gamma_\sigma s_\sigma + \delta_1 \omega_1(s_1) + \dots + \delta_\sigma \omega_\sigma(s_\sigma) \quad (182)$$

auf und bezeichnen die p Lösungen der Gleichung (180) mit $t_1, t_2, \dots t_p$, die denselben Combinationen der $u, v, s, \omega(s)$ entsprechenden Werthe der rechten Seite von (182) mit $T_1, T_2, \dots T_p$, so ist

$$t_1^n T_1 + t_2^n T_2 + \dots + t_p^n T_p,$$

wenn n eine positive ganze Zahl bedeutet, offenbar eine algebraische Function von $x, y, \ldots . y_1^{(m)}, y_2, \ldots . y_2^{(m)}, \ldots .$, welche für alle geschlossenen Wege der Variabeln x, welche $y_1, \ldots . y_1^{(m)}, y_2, \ldots . y_2^{(m)}, \ldots .$ zu ihrem Werthe zurückführen, stets unverändert bleibt, weil wenn t_α in t_β übergeht auch T_α in T_β übergeführt wird, es wird dieser Ausdruck somit eine rationale Function von $x, y_1, \ldots . y_1^{(m)}, y_2, \ldots . y_2^{(m)}, \ldots .$ sein, und somit, wenn $n = 0, 1, 2, \ldots . p - 1$ gesetzt wird, sich

$$\left.\begin{array}{l} T_1 + T_2 \quad + \ldots . + T_p \quad = f_0 \quad (x, y_1, \ldots . y_1^{(m)}, y_2, \ldots . y_2^{(m)}, \ldots .), \\ t_1 T_1 + t_2 T_2 \quad + \ldots . + t_p T_p \quad = f_1 \quad (x, y_1, \ldots . y_1^{(m)}, y_2, \ldots . y_2^{(m)}, \ldots .), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ t_1^{p-1} T_1 + t_2^{p-1} T_2 + \ldots . + t_p^{p-1} T_p = f_{p-1} (x, y_1, \ldots . y_1^{(m)}, y_2, \ldots . y_2^{(m)}, \ldots .), \end{array}\right\} (183)$$

ergeben, wenn $f_0, f_1, \ldots . f_{p-1}$ rationale Functionen bedeuten. Wenn nun

$$t^{p-1} + L_1 t^{p-2} + L_2 t^{p-3} + \ldots . + L_{p-1} = \psi(t) \tag{184}$$

gesetzt wird, und die Gleichungen (183) mit den noch unbestimmten Grössen $L_{p-1}, L_{p-2}, \ldots . L_1$, 1 multiplicirt und addirt werden, so folgt

$$T_1 \psi(t_1) + T_2 \psi(t_2) + \ldots . + T_p \psi(t_p) = L_{p-1} f_0 + L_{p-2} f_1 + \ldots . + f_{p-1}, \tag{185}$$

und wenn man

$$L_1 = \omega_1 + t_1,\ L_2 = \omega_2 + \omega_1 t_1 + t_1^2, \ldots . L_{p-1} = \omega_{p-1} + \omega_{p-2} t_1 + \ldots . + t_1^{p-1}, \tag{186}$$

setzt, so wird, wie aus (180) unmittelbar ersichtlich, $\psi(t)$ für $t_2, t_3, \ldots . t_{p-1}$ verschwinden, und für diese Werthe der L die Gleichung (185) den Ausdruck

$$T_1 = \frac{L_{p-1} f_0 + L_{p-2} f_1 + \ldots . + f_{p-1}}{t_1^{p-1} + L_1 t_1^{p-2} + \ldots . + L_{p-1}} \tag{187}$$

liefern, also T_1 als rationale, somit nach den Auseinandersetzungen am Anfange dieser Arbeit mit Hülfe von (180) als ganze Function $p - 1^{\text{ten}}$ Grades von t_1 darstellbar, deren Coefficienten rational aus $x, y_1, \ldots . y_1^{(m)}, y_2, \ldots . y_2^{(m)}, \ldots .$ zusammengesetzt sind. Stellt man also mit (180) $2\sigma + \rho$ der Gleichung (182) analoge Gleichungen zusammen, die sämmtlich mit constanten Coefficienten versehene lineare homogene Functionen der $u, v, s, \omega(s)$ sind, so werden deren linke Seiten, wie es mit T_1 in (182) der Fall war, sämmtlich ganze Functionen $p - 1^{\text{ten}}$ Grades derselben Grösse t_1 sein, deren Coefficienten rational aus $x, y_1, \ldots . y_1^{(m)}, y_2, \ldots . y_2^{(m)}, \ldots .$ zusammengesetzt sind, und somit wird sich aus den $2\sigma + \rho + 1$ in den $u, v, s, \omega(s)$ Grössen linearen Gleichungen (180) und (182) jede dieser

Grössen als eine ganze Function $p - 1^{\text{ten}}$ Grades von t_1 ausdrücken mit in den Grössen $x, y_1, \dots y_1^{(m)}, y_2, \dots y_2^{(m)}$ rationalen Coefficienten. Mögen diese rationalen Functionen mit

$$\mathfrak{a}(t),\ \mathfrak{b}(t),\ \mathfrak{c}(t),\ \mathfrak{d}(t)$$

bezeichnet werden, so werden den p Lösungen der Gleichung (180) die p Combinationen der Lösungen der Einzelgleichungen des Systemes (177) entsprechen

$$\left.\begin{array}{llll} u = \mathfrak{a}(t_1) & v_\delta = \mathfrak{b}_\delta(t_1) & s_\delta = \mathfrak{c}_\delta(t_1) & \omega_\delta(s_\delta) = \mathfrak{d}_\delta(t_1), \\ {}^{(1)}u = \mathfrak{a}(t_2) & {}^{(1)}v_\delta = \mathfrak{b}_\delta(t_2) & {}^{(1)}s_\delta = \mathfrak{c}_\delta(t_2) & {}^{(1)}\omega_\delta(s_\delta) = \mathfrak{d}_\delta(t_2), \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ {}^{(p-1)}u = \mathfrak{a}(t_p) & {}^{(p-1)}v_\delta = \mathfrak{b}_\delta(t_p) & {}^{(p-1)}s_\delta = \mathfrak{c}_\delta(t_p) & {}^{(p-1)}\omega_\delta(s_\delta) = \mathfrak{d}_\delta(t_p) \end{array}\right\} \quad (188)$$

Benutzen wir nun den eben bewiesenen Satz von der rationalen Darstellbarkeit aller Functionen $u, v, s, \omega(s)$ durch eine einzige durch die Gleichung (180) definirte algebraische Function, indem wir in der durch Differentiation der Gleichung (177) sich ergebenden Beziehung

$$y_1 = u' + A_1 \frac{v_1'}{v_1} + \dots + A_\rho \frac{v_\rho'}{v_\rho} + B_1 V_1(s_1) s_1' + \dots + B_\sigma V_\sigma(s_\sigma) s_\sigma', \quad (189)$$

in welcher $u', v_1', \dots v_\rho', s_1', \dots s_\sigma'$ resp. durch $u, v_1, \dots v_\rho, s_1, \dots s_\sigma$ als ganze Functionen ausdrückbar sind und $V_\alpha(s_\alpha)$ eine rationale Function von s_α und $\omega_\alpha(s_\alpha)$ ist, die durch das erste System von (188) gegebene Werthereihe durch t_1 ausgedrückt einsetzen. Es geht dann (189) offenbar in eine ganze Function von t_1 über, deren Coefficienten rationale Functionen von $x, y_1, \dots y_1^{(m)}, y_2, \dots y_2^{(m)}, \dots$ sind, und da die Gleichung (180) einerseits mit dieser die Lösung t_1 gemein hat, andererseits irreductibel ist, so wird jede Lösung von (180) auch (189) befriedigen oder es werden vermöge des Systemes (188) auch die Werthe ${}^{(\alpha)}u, {}^{(\alpha)}v, {}^{(\alpha)}s, {}^{(\alpha)}\omega(s)$ derselben genügen. Vermöge (177) werden also auch durch Uebergang zum Integral die Gleichungen bestehen

$$\left.\begin{array}{l} \int y_1 dx = u + A_1 \log v_1 + \dots + A_\rho \log v_\rho + B_1 \int^{s_1} V_1 ds + \dots + B_\sigma \int^{s_\sigma} V_\sigma ds, \\ \int y_1 dx = {}^{(1)}u + A_1 \log {}^{(1)}v_1 + \dots + A_\rho \log {}^{(1)}v_\rho + B_1 \int^{{}^{(1)}s_1} V_1 ds + \dots + B_\sigma \int^{{}^{(1)}s_\sigma} V_\sigma ds, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \int y_1 dx = {}^{(p-1)}u + A_1 \log {}^{(p-1)}v_1 + \dots + A_\rho \log {}^{(p-1)}v_\rho + B_1 \int^{{}^{(p-1)}s_1} V_1 ds \\ \qquad\qquad + \dots + B_\sigma \int^{{}^{(p-1)}s_\sigma} V_\sigma ds, \end{array}\right\} \quad (190)$$

wobei zu beachten, dass auch in den V-Grössen $\omega(s)$ durch ${}^{(\alpha)}\omega(s)$ zu ersetzen ist, und durch Addition, indem die linken Seiten der Gleichungen (190) sich nur um eine additive Constante unterscheiden können, und ${}^{(0)}u$, ${}^{(0)}v$, ${}^{(0)}s$, ${}^{(0)}\omega(s)$ die ursprünglichen Grössen u, v, s, $\omega(s)$ bedeuten sollen,

$$\int y_1 dx = \Sigma^{(\alpha)}u + A_1 \log \Pi^{(\alpha)}v_1 + \ldots . + A_\rho \log \Pi^{(\alpha)}v_\rho + B_1 \Sigma \int^{\overset{(\alpha)}{s_1}} V_1 ds$$
$$+ \ldots . + B_\sigma \Sigma \int^{\overset{(\alpha)}{s_\sigma}} V_\sigma ds, \quad (191)$$

wenn α die Werthe $0, 1, 2, \ldots . p-1$ annimmt.

Da nun zunächst $\Sigma^{(\alpha)}u$, $\Pi^{(\alpha)}v_1, \ldots . \Pi^{(\alpha)}v_\rho$ vermöge der Gleichungen (188) rationale symmetrische Functionen der Lösungen $t_1, t_2, \ldots . t_p$ der Gleichung (181) sind, so werden sich diese rational durch die Coefficienten derselben, also als rationale Functionen von $x, y_1, \ldots . y_1^{(m)}, y_2, \ldots . y_2^{(m)}, \ldots .$ darstellen lassen; betrachten wir ferner eine der Summen der Abel'schen Integrale, die wir ausführlicher in der Form schreiben wollen

$$\int^{s_\epsilon} V_\epsilon(s, \omega_\epsilon(s))\, ds + \int^{\overset{(1)}{s_\epsilon}} V_\epsilon(s, \omega_\epsilon(s))\, ds + \ldots . + \int^{\overset{(p-1)}{s_\epsilon}} V_\epsilon(s, \omega_\epsilon(s)\, ds, \quad (\mathrm{S})$$

so lässt sich diese bekanntlich nach dem Abel'schen Theorem von der Summe von Integralen algebraischer Functionen durch die Summe von π_ϵ gleichartigen Abel'schen Integralen und algebraisch-logarithmischen Functionen in der Form

$$\int^{\xi_\epsilon^{(1)}} V_\epsilon(s, \omega_\epsilon(s))\, ds + \int^{\xi_\epsilon^{(2)}} V_\epsilon(s, \omega_\epsilon(s))\, ds$$
$$+ \ldots . + \int^{\xi_\epsilon^{(\pi_\epsilon)}} V_\epsilon(s, \omega_\epsilon(s))\, ds + T_\epsilon + \sum_\beta A_{\epsilon\beta} \log T_\epsilon^{(\beta)} \quad (\mathrm{T})$$

ausdrücken, worin $A_{\epsilon\beta}$ Constanten, π_ϵ das Geschlecht der algebraischen Irrationalität $\omega_\epsilon(s)$ bedeutet, T_ϵ und $T_\epsilon^{(\beta)}$ rationale symmetrische Functionen der p oberen Integralgraenzen von (S) und der dazu gehörigen Irrationalitäten, also vermöge der Gleichungen (188) und (181) rational durch $x, y_1, \ldots . y_1^{(m)}$, $y_2, \ldots y_2^{(m)} \ldots$ ausdrückbar sind, ferner $\xi_\epsilon^{(1)}, \ldots \xi_\epsilon^{(\pi_\epsilon)}$ die Lösungen einer Gleichung $\pi_\epsilon^{\text{ten}}$ Grades vorstellen, deren Coefficienten wiederum rational und symmetrisch durch die Integralgraenzen von (S) und die dazu gehörigen Irrationalitäten also auch rational durch $x, y_1, \ldots . y_1^{(m)}, y_2, \ldots . y_2^{(m)}, \ldots .$ ausdrückbar sind, und die somit die Form hat

$$\xi_\epsilon^{\pi_\epsilon} + \Omega_1(x, y_1, \ldots . y_1^{(m)}, y_2, \ldots . y_2^{(m)}, \ldots .)\, \xi_\epsilon^{\pi_\epsilon - 1}$$
$$+ \ldots . + \Omega_{\pi_\epsilon}(x, y_1, \ldots . y_1^{(m)}, y_2, \ldots . y_2^{(m)}, \ldots .) = 0, \quad (192)$$

während die zu diesen Graenzen gehörigen Irrationalitäten, also die Werthe

$$\omega_\varepsilon(\xi_\varepsilon^{(1)}),\ \omega_\varepsilon(\xi_\varepsilon^{(2)}),\ \ldots.\ \omega_\varepsilon(\xi_\varepsilon^{(\pi_\varepsilon)})$$

rationale Functionen der resp. Graenzen und der Grössen $x, y_1, \ldots. y_1^{(m)}$, $y_2, \ldots. y_2^{(m)}, \ldots.$ sind. Fasst man nun das für die rechte Seite der Gleichung (191) entwickelte zusammen, so ergiebt sich der folgende Satz:

Ist die Quadratur eines Integrales y_1 einer algebraischen Differentialgleichung m^{ter} Ordnung im Abel'schen Sinne ausführbar, so ist dieselbe eine mit constanten Coefficienten versehene lineare Function logarithmischer und Abel'scher Transcendenten von der Form

$$\begin{aligned}\int y_1 dx = \mathfrak{U} + \mathfrak{A}_1 \log \mathfrak{V}_1 + \ldots. \mathfrak{A}_\mathfrak{r} \log \mathfrak{V}_\mathfrak{r} \\ + \mathfrak{B}_1 \left\{ \int^{\xi_1^{(1)}} V_1(s, \omega_1(s))\, ds + \ldots. + \int^{\xi_1^{(\pi_1)}} V_1(s, \omega_1(s)\, ds \right\} \\ + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + \mathfrak{B}_\sigma \left\{ \int^{\xi_\sigma^{(1)}} V_\sigma(s, \omega_\sigma(s))\, ds + \ldots. + \int^{\xi_\sigma^{(\pi_\sigma)}} V_\sigma(s, \omega_\sigma(s))\, ds \right\}, \end{aligned} \tag{193}$$

worin $\mathfrak{A}_1, \ldots. \mathfrak{A}_\mathfrak{r}$, $\mathfrak{B}_1, \ldots. \mathfrak{B}_\sigma$ *Constanten,* $\mathfrak{U}$, $\mathfrak{V}_1, \ldots. \mathfrak{V}_\mathfrak{r}$ *rationale Functionen* von x, $y_1, \ldots. y_1^{(m)}$, $y_2, \ldots. y_2^{(m)}, \ldots.$ *sind,* π_ε *das der algebraischen Function* $\omega_\varepsilon(s)$ *zugehörige Geschlecht bedeutet,* $\xi_\varepsilon^{(1)}$, $\xi_\varepsilon^{(2)}, \ldots. \xi_\varepsilon^{(\pi_\varepsilon)}$ *die Lösungen einer algebraischen Gleichung* π_ε^{ten} *Grades vorstellen, deren Coefficienten wiederum rational durch* x, $y_1, \ldots. y_1^{(m)}$, $y_2, \ldots. y_2^{(m)}, \ldots.$ *ausdrückbar sind, und für welche die zugehörigen Irrationalitäten mit Hülfe eben dieser Grössen durch die resp. Graenzen rational dargestellt werden können.*

Es ist nach den früheren Auseinandersetzungen wiederum klar, dass die rationalen Functionen von x, $y_1, \ldots. y_1^{(m)}$, $y_2, \ldots. y_2^{(m)}, \ldots.$ sich vermöge der Gleichung (1) oder (2) in ganze Functionen von $y_1^{(m)}$, $y_2^{(m)}, \ldots.$ vom $n-1^{ten}$ Grade verwandeln lassen, deren Coefficienten rationale Functionen von x, den Integralen und deren Ableitungen bis zur $m-1^{ten}$ Ordnung hin sind.

Da ferner die Differentiation der Gleichung (193) wiederum vermöge (1) oder (2) eine rationale Beziehung zwischen den Integralen und deren m ersten Ableitungen liefert, diese aber nach dem oben bewiesenen allgemeinen Hülfsatze erhalten bleibt bei der näher angegebenen passenden Substitution anderer Integrale, so ergiebt sich der Satz:

Wenn die Quadratur eines Integrales einer algebraischen Differentialgleichung m^{ter} Ordnung im Abel'schen Sinne ausführbar also in der Form (193) *darstellbar* *ist, so ist es auch die Quadratur eines jeden anderen Integrales eben dieser Differentialgleichung, welches ein beliebiges nicht singuläres Integral der in Bezug auf den höchsten Differentialquotienten algebraisch irreductibeln Differentialgleichung niedrigster Ordnung ist, welcher jenes Integral Genüge leistet, und zwar in genau derselben Form, wenn die anderen in dem Ausdrucke vorkommenden Integrale derselben Differentialgleichung passend gewählt sind.*

Soll die Reductionsformel der Quadratur auf der rechten Seite von (193) nur die Basis der Quadratur und nicht deren Ableitungen enthalten, so ergiebt sich durch Differentiation der Gleichung (193) unmittelbar, dass die Transcendente das Integral einer algebraischen Differentialgleichung erster Ordnung sein muss.

Zunächst können wir nun eine Eigenschaft der einzelnen Integralsummen, welche die rechte Seite der Gleichung (193) bilden, entwickeln; fassen wir nämlich den Coefficienten von $\mathfrak{B}_\epsilon$ in's Auge, welcher sich in der Form

$$\sum_1^{\pi_\epsilon}{}_\rho \int^{\xi_\epsilon^{(\rho)}} V_\epsilon(s, \omega(s))\, ds$$

darstellt, und dessen Graenzen die Lösungen der Gleichung (192) sind, so folgt durch Differentiation dieser letzteren Gleichung

$$d\xi_\epsilon^{(\rho)} = f(x, y_1, \dots y_1^{(m)}, y_2, \dots y_2^{(m)}, \dots, \xi_\epsilon^{(\rho)})\, dx, \tag{194}$$

worin f eine ganze Function $\pi_\epsilon - 1^{\text{ten}}$ Grades von ξ_ϵ bedeutet, deren Coefficienten rational aus $x, y_1, \dots y_1^{(m)}, y_2, \dots y_2^{(m)}, \dots$ zusammengesetzt sind, und wenn man nun diese Gleichung mit einer solchen rationalen Function von $\xi_\epsilon^{(\rho)}$ und $\omega_\epsilon(\xi_\epsilon^{(\rho)})$

$$W(\xi_\epsilon^{(\rho)}, \omega_\epsilon(\xi_\epsilon^{(\rho)}))$$

multiplicirt, dass die linke Seite der so resultirenden Gleichung

$$\begin{aligned} & W(\xi_\epsilon^{(\rho)}, \omega_\epsilon(\xi_\epsilon^{(\rho)}))\, d\xi_\epsilon^{(\rho)} \\ & \qquad = W(\xi_\epsilon^{(\rho)}, \omega_\epsilon(\xi_\epsilon^{(\rho)}))\, f(x, y_1, \dots y_1^{(m)}, y_2, \dots y_2^{(m)}, \dots \xi_\epsilon^{(\rho)})\, dx \end{aligned} \tag{195}$$

ein Abel'sches Differential erster Ordnung wird, so folgt, wenn auf beiden Seiten die Summe nach ρ von 1 bis π_ϵ genommen wird,

$$\begin{aligned} & \sum_1^{\pi_\epsilon}{}_\rho \int^{\xi_\epsilon^{(\rho)}} W(s, \omega_\epsilon(s))\, ds \\ & \qquad = \int \sum_1^{\pi_\epsilon}{}_\rho\, W(\xi_\epsilon^{(\rho)}, \omega_\epsilon(\xi_\epsilon^{(\rho)}))\, f(x, y_1, \dots y_1^{(m)}, y_2, \dots y_2^{(m)}, \dots, \xi_\epsilon^{(\rho)})\, dx. \end{aligned} \tag{196}$$

Da aber die Summe unter dem Integrale der rechten Seite dieser Gleichung eine rationale symmetrische Function von $\xi_\epsilon^{(1)}, \xi_\epsilon^{(2)}, \ldots . \xi_\epsilon^{(\pi_\epsilon)}$ ist, deren Coefficienten rational aus $x, y_1, \ldots . y_1^{(m)}, y_2, \ldots . y_2^{(m)}, \ldots .$ zusammengesetzt sind, sich also vermöge der Gleichung (192) als rationale Function der Grössen $x, y_1, \ldots . y_1^{(m)}, y_2, \ldots . y_2^{(m)}, \ldots .$

$$F(x, y_1, \ldots . y_1^{(m)}, y_2, \ldots . y_2^{(m)}, \ldots .)$$

darstellen lässt, so folgt

$$\sum_1^{\pi_\epsilon}{}_\rho \int^{\xi_\epsilon^{(\rho)}} W(s, \omega_\epsilon(s))\, ds = \int F(x, y_1, \ldots . y_1^{(m)}, y_2, \ldots . y_2^{(m)}, \ldots .)\, dx \qquad (197)$$

und wir erhalten daher den nachfolgenden Satz:

Ist die Quadratur eines Integrales einer algebraischen Differentialgleichung m^{ter} Ordnung im Abel'schen Sinne ausführbar, also in der Form (193) *darstellbar, so wird für jede Art von Abel'schen Integralen, die auf der rechten Seite von* (193) *vorkommen, die Summe eines jeden Systemes von Integralen* ERSTER *Gattung, deren Anzahl durch die den Integralen zukommende charakteristische Zahl π_ϵ bestimmt wird, und deren Graenzen durch die Gleichung* (192) *definirt sind, gleich einem Integrale, dessen Basis eine rationale Function der Integrale $y_1, y_2, \ldots .$ der gegebenen Differentialgleichung und deren Ableitungen bis zur m^{ten} Ordnung hin ist, oder anders ausgedrückt, dann giebt es immer eine Quadratur einer aus den Integralen und deren Ableitungen rational zusammengesetzten Function, welche gleich der Summe von π_ϵ Integralen erster Gattung ist, die zu irgend einer in der Reductionsformel* (193) *vorkommenden Art von Abel'schen Integralen gehören.*

Setzt man

$$F(x, y_1, \ldots . y_1^{(m)}, y_2, \ldots . y_2^{(m)}, \ldots . y_k, \ldots . y_k^{(m)}) = z$$

und differentiirt diese Gleichung $k.m$ mal nach x, indem man die höheren Ableitungen der Integrale als die m^{te} vermöge der Differentialgleichung (1) eliminirt, stellt diese $km + 1$ Gleichungen mit den k aus (1) hervorgehenden Gleichungen

$$y_1^{(m)} = f(x, y_1, y_1', \ldots . y_1^{(m-1)}),\ y_2^{(m)} = f(x, y_2, y_2', \ldots . y_2^{(m-1)}), \ldots .$$
$$y_\kappa^{(m)} = f(x, y_\kappa, y_\kappa', \ldots . y_k^{(m-1)})$$

zusammen und eliminirt aus diesen $k(m + 1) + 1$ Gleichungen die $k(m + 1)$ Grössen

$$y_1, y_1', \ldots . y_1^{(m)}, y_2, y_2', \ldots . y_2^{(m)}, \ldots . y_\kappa, y_\kappa', \ldots . y_k^{(m)},$$

so ergiebt sich für z die algebraische Differentialgleichung $\varkappa . m^{\text{ter}}$ Ordnung

$$\phi(x, z, z', \ldots . z^{(\varkappa m)}) = 0$$

und wir finden somit, dass

die Summe der eben aufgestellten π_ϵ Integrale erster Gattung im Allgemeinen der Quadratur eines Integrales einer algebraischen Differentialgleichung km^{ter} Ordnung gleich sein wird, und somit wieder gleich der Quadratur einer algebraischen Differentialgleichung m^{ter} Ordnung, wenn in F, also auch in der im Abel'schen Sinne ausführbaren Quadratur die algebraischen Functionen der rechten Seite von (171) *nur die Basis y_1 der Quadratur und deren Ableitungen enthalten.*

Bevor wir in der allgemeinen Untersuchung weitergehen, sollen zunächst einige Sätze über die zwischen Abel'schen Integralen stattfindenden Beziehungen oder über die Transformation Abel'scher Integrale entwickelt werden.

Man nennt ein Abel'sches Integral

$$\int^x y\,dx,$$

worin y eine zum Geschlechte p gehörige algebraische Function von x ist, reducirbar, wenn

$$\int^x y\,dx = F\left(x, \int^{u_1} y_1 ds, \int^{u_2} y_2 ds, \ldots . \int^{u_\rho} y_\rho ds\right) \tag{199}$$

ist, worin F eine algebraische Function der eingeschlossenen Grössen bedeutet, $y_1, y_2, \ldots . y_\rho$ algebraische Functionen von s vom Geschlechte $p_1, p_2, \ldots . p_\rho$, ferner $u_1, u_2, \ldots . u_\rho$ algebraische Functionen von x und $p_1, p_2, \ldots . p_\rho < p$ sind; ist das Geschlecht einzelner algebraischer Irrationalitäten Null, so gehen die zugehörigen Integrale in Logarithmen über.

Aus dem oben allgemein bewiesenen Satze für die im Abel'schen Sinne reducirbaren Quadraturen von Integralen beliebiger algebraischer Differentialgleichungen wird sich somit der Satz ergeben:

Ist ein Abel'sches Integral auf solche niederen Geschlechtes reducirbar, so ist dasselbe eine mit constanten Coefficienten versehene lineare Function logarithmischer Transcendenten und Abel'scher Integrale von der Form

$$\begin{aligned}\int y\,dx = \mathfrak{U} + \mathfrak{A}_1 \log \mathfrak{B}_1 + \ldots . + \mathfrak{A}_{\mathfrak{r}} \log \mathfrak{B}_{\mathfrak{r}} \\ + \mathfrak{B}_1\left\{\int^{\hat{\xi}_1^{(1)}} V_1(s, \omega_1(s))\,ds + \ldots . + \int^{\hat{\xi}_\sigma^{(\pi_1)}} V_1(s, \omega_1(s))\,ds\right\} \\ + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + \mathfrak{B}_\sigma\left\{\int^{\hat{\xi}^{(1)}} V_\sigma(s, \omega_\sigma(s))\,ds + \ldots . + \int^{\hat{\xi}_\sigma^{(\pi\sigma)}} V_\sigma(s, \omega_\sigma(s))\,ds\right\},\end{aligned} \tag{200}$$

worin $\mathfrak{A}_1, \dots . \mathfrak{A}_{\mathfrak{r}}, \mathfrak{B}_1, \dots . \mathfrak{B}$ *Constanten,* $\mathfrak{U}, \mathfrak{V}_1, \dots . \mathfrak{V}_{\mathfrak{r}}$ *rationale Functionen von* x *und* y *sind,* π_ϵ *das der algebraischen Function* $\omega_\epsilon(s)$ *zugehörige Geschlecht bedeutet,* $\xi_\epsilon^{(1)}, \xi_\epsilon^{(2)}, \dots . \xi_\epsilon^{(\pi_\epsilon)}$ *die Lösungen einer Gleichung* $\pi_\epsilon^{\text{ten}}$ *Grades vorstellen, deren Coefficienten wiederum rational durch* x *und* y *ausdrückbar sind, und für welche die zugehörigen Irrationalitäten mit Hülfe eben dieser Grössen durch die resp. Graenzen rational dargestellt werden können.*

Der eben ausgesprochene Satz wird sich, wie der Sinn der Herleitung desselben ergiebt, auch so ausdrücken lassen:

Jede zwischen Abel'schen Integralen mit algebraisch von einander abhängigen Graenzen existirende algebraische Beziehung ist eine mit constanten Coefficienten versehene lineare Relation von der Form (200).

Aber es tritt nunmehr die wichtige Frage auf, ob nicht eine algebraische Beziehung zwischen Abel'schen Integralen noch andere Functionen als Integrale algebraischer Functionen enthalten darf, wovon sich z. B. als ganz specieller Fall die Frage herleitem würde, ob nicht ein Abel'sches Integral mit Hülfe anderer Transcendenten ausführbar oder integrirbar sein könnte.

Nehmen wir zunächst an, es bestünde eine Relation der Form

$$\int y dx = F\left(x, \eta_1, \eta_2, \dots . \eta_r, \int^{u_1} y_1 ds, \int^{u_2} y_2 ds, \dots . \int^{u_\rho} y_\rho ds\right), \tag{201}$$

worin F eine algebraische Function der eingeschlossenen Grössen, $y_1, \dots . y_\rho$ algebraische Functionen von s, $u_1, u_2, \dots . u_\rho$ algebraische Functionen von x und $\eta_1, \eta_2, \dots . \eta_r$ transcendente Integrale der resp. algebraischen Differentialgleichungen erster Ordnung

$$\frac{d\eta}{dx} = f_1(\eta, x), \quad \frac{d\eta}{dx} = f_2(\eta, x), \dots . \frac{d\eta}{dx} = f_r(\eta, x) \tag{202}$$

sind, und werde angenommen, dass nicht schon zwischen den Argumenten von F selbst eine algebraische Beziehung bestehe.

Um die Möglichkeit der Existenz einer solchen Beziehung zu untersuchen, differentiire man (201) nach x und erhält

$$\begin{aligned} y = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial \eta_1} f_1(\eta_1, x) + \dots . + \frac{\partial F}{\partial \eta_r} f_r(\eta_r, x) \\ + \frac{\partial F}{\partial \int^{u_1} y_1 ds} (y_1)_{u_1} + \dots . + \frac{\partial F}{\partial \int^{u_\rho} y_\rho ds} (y_\rho)_{u_\rho}; \end{aligned} \tag{203}$$

da aber diese Relation eine algebraische Beziehung zwischen den Argumenten von F gegen die Voraussetzung liefern würde, so muss dieselbe in allen diesen Grössen identisch sein, und man wird daher in dieselbe z. B. statt η_1 jede Grösse also auch das allgemeine Integral der ersten der Differentialgleichungen (202) setzen dürfen, welches wir mit H_1 bezeichnen wollen; da dann die Integration der so entstehenden Gleichung (203)

$$\int y dx + c = F\left(x, H_1, \eta_2, \ldots . \eta_r, \int^{u_1} y_1 ds, \ldots . \int^{u_\sigma} y_\sigma ds\right) \tag{204}$$

liefert, so folgt durch Zusammenstellung von (201) und (204)

$$F\left(x, H_1, \eta_2, \ldots . \eta_r, \int^{u_1} y_1 ds, \ldots . \int^{u_\sigma} y_\sigma ds\right)$$
$$= F\left(x, \eta_1, \eta_2, \ldots . \eta_r, \int^{u_1} y_1 ds, \ldots . \int^{u_\sigma} y_\sigma ds\right) + c, \tag{205}$$

worin c eine Constante bedeutet.

Nehmen wir nun an, dass die Differentialgleichung erster Ordnung

$$\frac{d\eta}{dx} = f_1(\eta, x) \tag{206}$$

die Eigenschaft habe, dass das allgemeine Integral H_1 eine algebraische Function des particulären Integrales η_1 also

$$H_1 = F_1(x, \eta_1, \varkappa) \tag{207}$$

sei,* so ginge die Gleichung (205) in

$$F\left(x, F_1(x, \eta_1, \varkappa), \eta_2, \ldots . \eta_r, \int^{u_1} y_1 ds, \ldots . \int^{u_\sigma} y_\sigma ds\right)$$
$$= F\left(x, \eta_1, \eta_2, \ldots . \eta_r, \int^{u_1} y_1 ds, \ldots . \int^{u_\sigma} y_\sigma ds\right) + c \tag{208}$$

über und muss, da eine algebraische Beziehung zwischen

$$\eta_1, \eta_2, \ldots . \eta_r, \int^{u_1} y_1 ds, \ldots . \int^{u_\sigma} y_\sigma ds$$

ausgeschlossen war, eine in allen diesen Grössen identische sein, wobei c eine von der willkührlichen Grösse $\varkappa$ abhängige Constante sein wird.

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* Wenn in der Gleichung (201) die rechte Seite nur eine algebraische Function von x und η_1 wäre, also die Gleichung (205)

$$F(x, H_1) = F(x, \eta_1) + c$$

lautete, so würde aus dieser Gleichung schon von selbst folgen, dass H_1 eine algebraische Function von η_1 wäre.

Differentiirt man nun die Gleichung (208) nach η_1 und k, so ergiebt sich

$$\left.\begin{aligned} \frac{\partial F(x, F_1, \eta_2, \ldots .)}{\partial F_1} \frac{\partial F_1(x, \eta_1, k)}{\partial \eta_1} &= \frac{\partial F(x, \eta_1, \ldots .)}{\partial \eta_1}, \\ \frac{\partial F(x, F_1, \eta_2, \ldots .)}{\partial F_1} \frac{\partial F_1(x, \eta_1, k)}{\partial k} &= \frac{\partial c}{\partial k} \end{aligned}\right\} \tag{209}$$

und hieraus

$$\frac{\partial F\left(x, \eta_1, \eta_2, \ldots . \eta_r, \int^{u_1} y_1 dx, \ldots . \int^{u_\sigma} y_\sigma ds\right)}{\partial \eta} = \frac{\partial c}{\partial k} \frac{\dfrac{\partial F_1(x, \eta_1, k)}{\partial \eta_1}}{\dfrac{\partial F_1(x, \eta_1, k)}{\partial k}}$$

als eine wieder in allen bezeichneten Grössen identische Gleichung und daher

$$F\left(x, \eta_1, \ldots . \eta_r, \int^{u_1} y_1 ds, \ldots . \int^{u_\sigma} y_\sigma ds\right) = \phi(x, \eta_1) + L, \tag{210}$$

worin ϕ eine algebraische Function sein muss und L die Grösse η_1 nicht mehr enthält. Nehmen wir nun an, dass für alle Differentialgleichungen (202) das allgemeine Integral eine algebraische Function eines particulären Integrales der resp. Differentialgleichung ist, und bemerken, dass die Integrale

$$\int^{u_\alpha} y_\alpha ds = \xi_\alpha$$

ebenfalls Integrale von Differentialgleichungen erster Ordnung von der Form

$$\frac{d\xi_\alpha}{dx} = (y_\alpha)_{u_\alpha} u'_\alpha \tag{211}$$

sind, für welche der algebraische Zusammenhang zwischen dem allgemeinen und einem particulären Integrale

$$\xi = \xi_\alpha + k$$

lautet, so folgt nach (210), dass die Form von F lautet*

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* Setzt man nämlich in (208) statt $\int^{u_1} y_1 ds$ die Grösse $\int^{u_1} y_1 ds + k$, so erhält man

$$\int y dx = F\left(x, \eta_1, \ldots . \eta_r, \int^{u_1} y_1 ds + k, \ldots . \int^{\sigma} y_\sigma ds\right)$$

und somit statt (205) die Beziehung

$$F\left(x, \eta_1, \ldots . \eta_r, \int^{u_1} y_1 ds + k, \ldots . \int^{u_\sigma} y_\sigma ds\right) = F\left(x, \eta_1, \ldots \eta_r, \int^{u_1} y_1 ds, \ldots . \int^{u_\sigma} y_\sigma ds\right) + c,$$

$$F\left(x, \eta_1, \eta_2, \ldots . \eta_r, \int^{u_1} y_1 ds, \ldots . \int^{u_\sigma} y_\sigma ds\right)$$
$$= \phi_1(x, \eta_1) + \phi_2(x, \eta_2) + \ldots . + \phi_r(x, \eta_r) + A_1 \int^{u_1} y_1 ds + \ldots . + A_\sigma \int^{u_\sigma} \eta_\sigma ds \quad (212)$$

oder nach (201)

$$\int y dx = \phi_1(x, \eta_1) + \phi_2(x, \eta_2) + \ldots + \phi_r(x, \eta_r) + A_1 \int^{u_1} y_1 ds + \ldots + A_\sigma \int^{u_\sigma} y_\sigma ds, \quad (213)$$

worin die ϕ algebraische Functionen und die A Constanten bedeuten.

Setzt man nun

$$\phi_\lambda(x, y_r) = \xi_r \quad (214)$$

und transformirt mit Hülfe dieser Substitution die Differentialgleichung

$$\frac{d\eta_r}{dx} = f_r(\eta_r, x) \quad (215)$$

in die Differentialgleichung erster Ordnung

$$\frac{d\xi_r}{dx} = \omega_r(\xi_r, x), \quad (216)$$

für welche der dem angenommenen Zusammenhange

$$H_r = F_r(x, \eta_r, k) \quad (217)$$

vermöge (214) entsprechende algebraische Zusammenhang zwischen dem allgemeinen und einem particulären Integrale

$$\Xi_r = \Phi_r(x, \xi_r, k) \quad (218)$$

lauten möge, so wird die Beziehung (213) in

$$\int y dx = \xi_1 + \xi_2 + \ldots . + \xi_r + A_1 \int^{u_1} y_1 ds + \ldots . + A_\sigma \int^{u_\sigma} y_\sigma ds \quad (219)$$

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woraus durch Differentiation nach $\int^{u_1} y_1 ds$ und k sich

$$\frac{\partial F\left(x, \eta_1, \ldots . \eta_r, \int^{u_1} y_1 ds + k, \ldots . \int^{u_\sigma} y_\sigma ds\right)}{\partial\left(\int^{u_1} y_1 ds + k\right)} = \frac{\partial F\left(x, \eta_1, \ldots . \eta_r, \int^{u_1} y_1 ds, \ldots . \int^{u_\sigma} y_\sigma ds\right)}{\partial \int^{u_1} y_1 ds} = \frac{dc}{dk},$$

also

$$F\left(x, \eta_1, \ldots . \eta_r, \int^{u_1} y_1 ds, \ldots . \int^{u_\sigma} y_\sigma ds\right) = A_1 \int^{u_1} y_1 ds + M$$

folgt, worin M von $\int^{u_1} y_1 ds$ unabhängig ist.

übergehen, und daher genau nach den früheren Schlüssen

$$\Phi_r(x, \xi_r, k) = \xi_r + c \tag{220}$$

und nach (218)

$$\Xi_r = \xi_r + c;$$

daraus folgt aber vermöge (216)

$$\omega_r(\xi_r + c, x) = \omega_r(\xi_r, x)$$

und somit, weil c beliebig ist,

$$\omega_r(\xi_r, x) = \Omega_r(x), \tag{221}$$

worin $\Omega_r(x)$ eine algebraische Function bedeutet und vermöge (216)

$$\xi_r = \int \Omega_r(x)\, dx$$

ein Abel'sches Integral, also nach (214) η_r eine algebraische Function von x und einem Abel'schen Integrale ist, so dass (219) in

$$\int y dx = \int \Omega_1(x)\, dx + \ldots. + \int \Omega_r(x)\, dx + A_1 \int^{u_1} y_1 ds + \ldots. + A_\sigma \int^{u_\sigma} y_\sigma ds \tag{222}$$

übergeht, die Relation also nur Abel'sche Integrale und zwar in linearer Form enthalten darf; selbstverständlich können die letzten Integrale auch in Abel'sche Integrale mit der oberen Graenze x umgesetzt werden.

Wir erhalten somit den folgenden Satz:

In eine algebraische Beziehung zwischen Abel'schen Integralen können nie andere Integrale von solchen algebraischen Differentialgleichungen erster Ordnung eintreten, deren allgemeines Integral von einem particulären Integrale algebraisch abhängt,

oder anders ausgedrückt,

Die algebraische Reduction eines Abel'schen Integrales ist nie durch Integrale von solchen algebraischen Differentialgleichungen erster Ordnung möglich, deren allgemeines Integral eine algebraische Function eines ihrer particulären Integrale ist.

So würden also in eine solche Beziehung nie die Exponentialfunction, die elliptischen Functionen, kurz keine Function eintreten dürfen, die ein Additionstheorem besitzt; denn hat $\eta = \psi(x)$ ein Additionstheorem, so dass

$$\psi(x + y) = \omega(\psi(x), \psi(y)), \tag{223}$$

ist, worin ω eine algebraische Function bedeutet, so folgt durch Differentiation nach x und y

$$\psi'(x + y) = \frac{\partial \omega}{\partial \psi(x)} \psi'(x) = \frac{\partial \omega}{\partial \psi(y)} \psi'(y),$$

und es ist somit, indem man für y irgend einen constanten Werth gesetzt denkt, $X = \psi(x)$ das Integral einer algebraischen Differentialgleichung erster Ordnung

$$\Omega(\psi(x), \psi'(x)) = 0 \quad X' = f(X), \tag{224}$$

welche von der unabhängigen Variabeln frei ist; da aber hieraus unmittelbar folgt, dass auch $\psi(x + c)$ ein Integral also das allgemeine Integral von (224) ist, und ausserdem nach dem Additionstheorem (223)

$$\psi(x + c) = \omega(\psi(x), \psi(c)) \tag{225}$$

ist, so ist die Bedingung erfüllt, dass das allgemeine Integral eine algebraische Function eines particulären ist und es kann somit keine solche Function in der Reductionsformel eines Abel'schen Integrales vorkommen.

Wenn das Abel'sche Integral auf eine Reihe Abel'scher Integrale und *ein* Integral einer algebraischen Differentialgleichung erster Ordnung reducirbar wäre, also die Gleichung (201) die Form hätte

$$\int y dx = F\left(x, \eta_1, \int^{u_1} y_1 ds, \ldots . \int^{u_\sigma} y_\sigma ds\right), \tag{226}$$

so würden nach den früheren Auseinandersetzungen die Abel'schen Integrale der rechten Seite in linearer Form mit constanten Coefficienten vorkommen müssen, und da dann die Gleichung (226) die Gestalt annähmd

$$\int Y dx = \phi(x, \eta_1), \tag{227}$$

indem die additiv verbundenen Abel'schen Integrale in eins zusammengefasst sind, so würde durch Differentiation sich wieder

$$Y = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial \eta_1} f(\eta_1, x)$$

ergeben, und da η_1 keine algebraische Function sein soll, also diese Gleichung eine identische sein muss, wieder durch Integration, wenn H_1 das allgemeine Integral der Differentialgleichung erster Ordnung bezeichnet,

$$\phi(x, H_1) = \phi(x, \eta_1) + c$$

also das allgemeine Integral eine algebraische Function eines particulären Integrales sein müssen, welcher Fall für die Existenz der Gleichung (227) als unstatthaft erwiesen war; es folgt somit, *dass zwischen Abel'schen Integralen und* EINEM *Integrale einer algebraischen Differentialgleichung erster Ordnung ein algebraischer Zusammenhang nicht stattfinden kann.*

Betrachten wir endlich den allgemeinen Fall der Beziehung (201), worin $\eta_1, \ldots. \eta_r$ Integrale von algebraischen Differentialgleichungen erster Ordnung sein sollen, so musste die Gleichung (203) eine identische sein und somit durch Substitution von $\int^{u_\alpha} y_\alpha ds + \lambda$ statt $\int^{u_\alpha} y_\alpha ds$, wenn λ eine willkührliche Constante bedeutet, durch Integration und Zusammenstellung mit (201) sich

$$F\left(x, \eta_1, \ldots. \eta_r, \int^{u_1} y_1 ds, \ldots. \int^{u_\alpha} y_\alpha ds + \lambda, \ldots. \int^{u_\sigma} y_\sigma ds\right)$$
$$= F\left(x, \eta_1, \ldots. \eta_r, \int^{u_1} y_1 ds, \ldots. \int^{u_\alpha} y_\alpha ds, \ldots. \int^{u_\sigma} y_\sigma ds\right) + c \qquad (228)$$

ergeben, welche wieder der gemachten Annahme gemäss eine identische sein muss, woraus nach wiederholt dagewesenen Schlüssen

$$F\left(x, \eta_1, \ldots. \eta_r, \int^{u_1} y_1 ds, \ldots. \int^{u_\sigma} y_\sigma ds\right)$$
$$= \omega(x, \eta_1, \ldots. \eta_r) + A_1 \int^{u_1} y_1 ds + \ldots. + A_\sigma \int^{u_\sigma} y_\sigma ds \qquad (229)$$

folgt, wenn ω eine algebraische Function und $A_1, \ldots. A_\sigma$ Constanten bedeuten. Da nun aus (201)

$$\int y dx = \omega(x, \eta_1, \ldots. \eta_r) + A_1 \int^{u_1} y_1 ds + \ldots. + A_\sigma \int^{u_\sigma} y_\sigma ds \qquad (230)$$

sich ergiebt, ausserdem die Abel'schen Integrale in ein solches

$$\int^x \left(y - A_1 (y_1)_{u_1} u_1' - \ldots - A_\sigma (y_\sigma)_{u_\sigma} u_\sigma'\right) dx = \int Y dx$$

zusammengefasst werden können, so müsste

$$\int Y dx = \omega(x, \eta_1, \ldots. \eta_r) \qquad (231)$$

sein; da aber durch Differentiation

$$Y = \frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial \eta_1} f_1(\eta_1, x) + \ldots. + \frac{\partial \omega}{\partial \eta_r} f_r(\eta_r, x)$$

folgt und diese Gleichung wieder eine identische sein muss also auch bestehen bleibt, wenn $\eta_1, \ldots. \eta_r$ durch die allgemeinen Integrale $H_1, H_2, \ldots. H_r$ ihrer resp. Differentialgleichungen ersetzt werden, so wird wieder durch Integration und Vergleichung mit (231)

$$\omega(x, H_1, H_2, \dots . H_r) = \omega(x, \eta_1, \eta_2, \dots . \eta_r) + c \tag{232}$$

oder

$$\int Ydx + c = \omega(x, H_1, H_2, \dots . H_r) \tag{233}$$

folgen; lassen wir nun $\eta_2, \dots . \eta_r$ unverändert, so dass aus (233)

$$\int Ydx + k = \omega(x, H_1, \eta_2, \dots . \eta_r) \tag{234}$$

folgt, so würde sich nach (231) und (234) H_1 und η_1 durch dieselben Functionen $\eta_2, \dots . \eta_r, \int Ydx$ algebraisch in derselben Weise ausdrücken lassen, also alle Integrale der Differentialgleichung

$$\frac{\partial \eta_1}{dx} = f_1(\eta_1, x)$$

durch denselben algebraischen Ausdruck derselben Functionen dargestellt sein, wenn nur die additive Constante bei $\int Ydx$ entsprechend geändert wird; dies kann aber im Allgemeinen für Differentialgleichungen erster Ordnung nicht stattfinden, deren Integrale verschiedene selbständige Transcendente darstellen, wenn nicht ihr allgemeines Integral eine algebraische Function von particulären Integralen ist. Es ergiebt sich somit der Satz:

Zwischen Abel'schen Integralen und Integralen von algebraischen Differentialgleichungen erster Ordnung kann nie ein algebraischer Zusammenhang stattfinden.

Es braucht kaum hervorgehoben zu werden, dass die angewandten Betrachtungen, auch wenn die η nicht algebraischen Differentialgleichungen erster Ordnung angehören, häufig zu einfachen Resultaten führen. Sei z. B. die Reduction eines Abel'schen Integrales auf andere solche Integrale und auf das Integral und dessen Ableitungen irgend einer algebraischen Differentialgleichung beliebiger Ordnung

$$\eta^{(m)} = f(x, \eta, \eta', \dots . \eta^{(m-1)}) \tag{235}$$

zu untersuchen, und werde diese Reductionsformel dargestellt durch

$$\int ydx = F\left(x, \eta_1, \eta_1', \dots . \eta_1^{(m-1)}, \int^{u_1} y_1 ds, \dots . \int^{u_\sigma} y_\sigma ds\right), \tag{236}$$

so darf zunächst angenommen werden, dass zwischen den Argumenten der algebraischen Function F ein algebraischer Zusammenhang nicht stattfindet, weil man sonst die entsprechende Anzahl Abel'scher Integrale aus der rechten Seite von (236) herausschaffen und die so transformirte Reductionsformel der weiteren Untersuchung zu Grunde legen würde. Da nun die Differentiation von (236) mit Benutzung von (235) die Beziehung

$$y = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial \eta_1}\eta_1' + \dots + \frac{\partial F}{\partial \eta_1^{(m-1)}} f(x, \eta_1, \eta_1^{(m-1)})$$
$$+ \frac{\partial F}{\partial \int^{u_1} y_1 ds}(y_1)_{u_1} u_1' + \dots + \frac{\partial F}{\partial \int^{u_\sigma} y_\sigma ds}(y_\sigma)_{u_\sigma} u_\sigma'$$

liefert und diese der gemachten Annahme gemäss identisch sein muss, so wird, wenn H_1 irgend ein anderes Integral von (235) bedeutet und $k_1, k_2, \dots k_\sigma$, c Constanten sind,

$$\int y dx + c = F\Big(x, H_1, H_1', \dots H_1^{(m-1)}, \int^{u_1} y_1 ds + k_1, \dots \int^{u_\sigma} y_\sigma ds + k_\sigma \tag{237}$$

sein, woraus zunächst wieder nach wiederholt dagewesenen Schlüssen

$$\int y dx + C = A_1 \int^{u_1} y_1 ds + A_2 \int^{u_2} y_2 ds + \dots + A_\sigma \int^{u_\sigma} y_\sigma ds + \omega(x, H_1, H_1', \dots H_1^{(m-1)}) \tag{238}$$

oder

$$\int Y dx + C = \omega(x, H_1, H_1', \dots H_1^{(m-1)}) \tag{239}$$

folgt. Da nun

$$\int Y dx + C_1 = \omega(x, \eta_1, \eta_1', \dots \eta_1^{(m-1)}) \tag{240}$$

ist, so wird durch Zusammenstellung mit (239)

$$\omega(x, H_1, H_1', \dots H_1^{(m-1)}) = \omega(x, \eta_1, \eta_1', \dots \eta_1^{(m-1)}) + K, \tag{241}$$

worin K eine Constante bedeutet; machen wir nun für die Differentialgleichung m^{ter} Ordnung (235) die Annahme, dass das Integral η_1 nicht schon einer algebraischen Differentialgleichung niederer Ordnung als der m^{ten} genogt, und dass dieselbe ferner die Eigenschaft hat, dass auch $\mu\eta_1$, worin μ ein willkührlicher constanter Multiplicator ist, ein Integral von (235) ist, so wird die aus (241) entspringende Relation

$$\omega(x, \mu\eta_1, \mu\eta_1', \dots \mu\eta_1^{(m-1)}) = \omega(x, \eta_1, \eta_1' \dots \eta_1^{(m-1)}) + K_1 \tag{242}$$

eine in allen Grössen identische sein müssen. Differentiirt man dieselbe nach $\eta_1, \eta_1', \dots \eta_1^{(m-1)}$ und μ, so folgt

$$\frac{\partial\omega(x, \mu\eta_1, \dots \mu\eta_1^{(m-1)})}{\partial\mu\eta_1}\mu = \frac{\partial\omega(x, \eta_1, \dots \eta_1^{(m-1)})}{\partial\eta_1}, \dots \frac{\partial\omega(x, \mu\eta_1, \dots \mu\eta_1^{(m-1)})}{\partial\mu\eta_1^{(m-1)}}\mu$$
$$= \frac{\partial\omega(x, \eta_1, \dots \eta_1^{(m-1)})}{\partial\eta_1^{(m-1)}}, \quad \frac{\partial\omega(x, \mu\eta_1, \dots)}{\partial\mu\eta_1}\eta_1 + \dots + \frac{\partial\omega(x, \mu\eta_1, \dots)}{\partial\mu\eta_1^{(m-1)}}\eta_1^{(m-1)} = \frac{\partial K_1}{\partial\mu}$$

und hieraus die Gleichung

$$\eta_1 \frac{\partial \omega(x, \eta_1, \ldots . \eta_1^{(m-1)})}{\partial \eta_1} + \eta_1' \frac{\partial \omega(x, \eta_1, \ldots . \eta_1^{(m-1)})}{\partial \eta_1'} + \ldots . \\ + \eta_1^{(m-1)} \frac{\partial \omega(x, \eta_1, \ldots . \eta_1^{(m-1)})}{\partial \eta_1^{(m-1)}} = \mu \frac{\partial K_1}{\partial \mu},$$

die wiederum in $\eta_1, \eta_1', \ldots . \eta_1^{(m-1)}$ identisch sein muss. Diese partielle Differentialgleichung führt auf das System totaler Differentialgleichungen

$$\frac{d\eta_1}{\eta_1} = \frac{d\eta_1'}{\eta_1'} = \ldots . = \frac{d\eta_1^{(m-1)}}{\eta_1^{(m-1)}} = \frac{d\omega}{\mu \frac{\partial K_1}{\partial \mu}},$$

woraus

$$\frac{\eta_1'}{\eta_1} = \alpha_1, \quad \frac{\eta_1''}{\eta_1} = \alpha_2, \ldots . \frac{\eta_1^{(m-1)}}{\eta_1} = \alpha_{m-1}, \quad \omega = \mu \frac{\partial K_1}{\partial \mu} \log \eta_1 + \beta,$$

also

$$\omega = \mu \frac{\partial K_1}{\partial \mu} \log \eta_1 + \phi\left(x, \frac{\eta_1'}{\eta_1}, \frac{\eta_1''}{\eta_1}, \ldots . , \frac{\eta_1^{(m-1)}}{\eta_1}\right) \tag{243}$$

folgt. Da aber ω eine algebraische Function von $\eta_1, \eta_1', \ldots . \eta_1^{(m-1)}$ sein soll, so muss $\frac{\partial K_1}{\partial \mu} = 0$ sein und somit die Gleichung (240) in

$$\int Y dx + C_1 = \phi\left(x, \frac{\eta_1'}{\eta_1}, \frac{\eta_1''}{\eta_1}, \ldots . \frac{\eta_1^{(m-1)}}{\eta_1}\right) \tag{244}$$

übergehen, eine Reductionsformel, wie sie z. B. stets für ein Integral η_1 einer linearen homogenen irreductibeln Differentialgleichung stattfinden müsste. Aus (244) kann man z. B. schliessen, *dass zwischen Abel'schen Integralen und der Function*

$$\eta = e^{e^x}, \tag{245}$$

welche der Differentialgleichung

$$\eta\eta'' - \eta'^2 - \eta\eta' = 0 \tag{246}$$

genügt, eine algebraische Beziehung nie stattfinden kann; denn da einerseits e^{e^x} nicht schon einer algebraischen Differentialgleichung erster Ordnung genügt, andererseits auch μe^{e^x} ein Integral der Differentialgleichung (246) ist, so würde die etwa bestehende Reductionsformel nach (244) lauten müssen

$$\int Y dx + C_1 = \phi(x, e^x)$$

und dies ist nach dem Früheren unmöglich, da Exponentialgrössen nicht in eine algebraische Beziehung zwischen Abel'schen Integralen eintreten können, und ähnliche Resultate kann man für die aus anderen Functionen, welche algebraische Additionstheoreme haben, iterirten Transcendenten herleiten.

Nach Ausführung dieser Untersuchungen für Abel'sche Integrale wird auch der Weg vorgezeichnet sein, auf dem man bei der Untersuchung der Quadraturen von Integralen algebraischer Differentialgleichungen fortzuschreiten hat, wenn man nicht bloss wie oben in Gleichung (171) eine Reduction auf Abel'sche Integrale, deren Graenzen algebraisch aus der Basis der Quadratur und deren Ableitungen sowie anderer Integrale derselben Differentialgleichung und deren Ableitungen, voraussetzt, sondern analog der oben für Abel'sche Integrale definirten Reduction auch in diesem allgemeinen Falle die Reduction folgendermassen definirt.

Wenn y_1 ein Integral einer algebraischen Differentialgleichung m^{ter} Ordnung

$$y^{(m)^n} + F_1(x, y, y', \dots y^{(m-1)})\, y^{(m)^{n-1}} + \dots + F_n(x, y, y', \dots y^{(m-1)}) = 0 \tag{247}$$

ist, worin $F_1, \dots F_n$ rationale Functionen bedeuten und welche in Bezug auf den höchsten Differentialquotienten $y^{(m)}$ mit Adjungirung von $x, y, y', \dots y^{(m-1)}$ als irreductibel im algebraischen Sinne vorausgesetzt wird, so soll, wenn y_1 ein Integral dieser Differentialgleichung bedeutet, welches nicht schon einer algebraischen Differentialgleichung von niederer Ordnung als der m^{ten} Genüge leistet, die Quadratur dieses Integrales reductibel genannt werden, wenn*

$$\int y_1 dx = F(x, y_1, y_1', \dots y_1^{(m)}, y_2, y_2', \dots y_2^{(m)}, \dots \eta_1(u_1), \eta_1'(u_1), \dots \eta_2(u_2), \eta_2'(u_2), \dots) \tag{248}$$

ist, worin F eine algebraische Function, $y_1, y_2, \dots$ Integrale der Differentialgleichung (247), $u_1, u_2, \dots$ algebraische Functionen von $y_1, y_1', \dots y_2, y_2', \dots$

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* Es ist leicht einzusehen, dass y_1 nicht schon einer algebraischen Differentialgleichung m^{ter} Ordnung und niedrigeren Grades als vom n^{ten} genügen kann; denn wäre dies der Fall und diese Differentialgleichung

$$y^{(m)^\nu} + f_1(x, y, y', \dots y^{(m-1)})\, y^{(m)^{\nu-1}} + \dots + f_\nu(x, y, y', \dots y^{(m-1)}) = 0, \tag{a}$$

so verfahre man zwischen den in $y^{(m)}$ ganzen Polynomen des n^{ten} und ν^{ten} Grades (247) und (a) nach der Methode der Aufsuchung des grössten gemeinschaftlichen Theilers, dann ist klar, dass der jedesmalige Rest, wenn y_1 statt y gesetzt wird, verschwinden muss, da Dividendus und Divisor für $y = y_1$ zu Null werden, und die Division darf keinen den beiden Polynomen (247) und (a) gemeinsamen Theiler ergeben, da (247) in $y^{(m)}$ als irreductibel in algebraischem Sinne vorausgesetzt wurde, somit bleibt also ein Rest, welcher eine rationale Function von $x, y, y', \dots y^{(m-1)}$ ist und aus den angegebenen Gründen für $y = y_1$ verschwinden muss. Da aber y_1 der Annahme nach keiner algebraischen Differentialgleichung von niederer Ordnung als der m^{ten} Genüge leisten sollte, so ist der Widerspruch nur so zu lösen, dass y_1 nicht einer Gleichung der Form (a) von der m^{ten} Ordnung und niedrigeren Grades als dem n^{ten} genügen kann. Nimmt man umgekehrt an, dass y_1 nicht einer algebraischen Differentialgleichung m^{ter} Ordnung und niederen Grades als dem n^{ten} genügen darf, so folgt die algebraische Irreductibilität der Gleichung (247) in Bezug auf $y^{(m)}$ von selbst, da eine Zerlegung auch eine solche für y_1 nach sich ziehen würde.

und endlich $\eta_1, \eta_2, \ldots.$ *Integrale algebraischer Differentialgleichungen von niedrigerer Ordnung als der* m^{ten} *oder von der* m^{ten} *Ordnung aber niedrigeren Grades als dem* n^{ten} *bedeuten.*

Um zu zeigen, in welcher Weise die oben hergeleiteten Sätze für die Untersuchung solcher reducirbarer Quadraturen von Integralen algebraischer Differentialgleichungen benutzt werden können, wollen wir unter der Annahme, dass y_1 ein Integral der Differentialgleichung (247) ist, die Reductionsformel

$$\int y_1 dx = F(x, y_1, y_1', \ldots y_1^{(m)}, y_2, y_2', \ldots y_2^{(m)}, \ldots \eta_1(u_1), \eta_1'(u_1), \ldots \eta_1^{\mu}(u_1)) = z_1 \quad (249)$$

untersuchen, in welcher $\eta_1(u_1)$ ein Integral der algebraischen Differentialgleichung

$$\eta^{(\mu)^\nu} + \phi_1(u_1, \eta, \eta', \ldots . \eta^{(\mu-1)}) \eta^{(\mu)^{\nu-1}} + \ldots . + \phi_\nu(u_1, \eta, \eta', \ldots . \eta^{(\mu-1)}) = 0 \quad (250)$$

ist, worin $\mu < m$, und u_1 die Lösung der algebraischen Gleichung

$$u^\lambda + \psi_1(x, y_1, y_1', \ldots y_2, y_2', \ldots) u^{\lambda-1} + \ldots + \psi_\lambda(x, y_1, y_1', \ldots y_2, y_2', \ldots) = 0, \quad (251)$$

welche als eine mit Adjungirung von $y_1, y_2, \ldots.$ und deren Ableitungen irreductible vorausgesetzt werden darf.

Sei z_1 oder die F-function durch die algebraische Gleichung definirt

$$z^k + \omega(x, y_1, y_1', \ldots . y_2, y_2', \ldots ., u_1, \eta_1(u_1), \eta_1'(u_1), \ldots .) z^{k-1} + \ldots .$$
$$+ \omega_k(x, y_1, y_1', \ldots . y_2, y_2', \ldots ., u_1, \eta_1(u_1), \eta_1'(u_1), \ldots .) = 0 \quad (252)$$

worin $\omega_1, \ldots . \omega_k$ rationale Functionen der eingeschlossenen Grössen bedeuten, und welche mit Adjungirung eben dieser Grössen irreductibel angenommen werden darf, so ergiebt sich nach am Anfange dieser Untersuchung ausgeführten Rechnungen, da

$$\frac{d}{dx}(\eta_1^{(r)}(u_1)) = \eta_1^{(r+1)}(u_1) u_1',$$

und u_1' nach Gleichung (251) sich als ganze Function $\lambda - 1^{ten}$ Grades von u_1 ergeben,

$$z' = P_0 z^{k-1} + P_1 z^{k-2} + \ldots . + P_{k-1}, \quad (253)$$

worin $P_0, P_1, \ldots . P_{k-1}$ rationale Functionen von

$$x, y_1, y_1', \ldots . y_1^{(m)}, y_2, y_2', \ldots . y_2^{(m)} \ldots . u_1, \eta_1(u_1), \eta_1'(u_1), \ldots . \eta_1^{(\mu)}(u_1)$$

bedeuten. Da aber nach (249) $z_1' = y_1$ ist, so geht (253) in

$$P_0 z_1^{k-1} + P_1 z_1^{k-2} + \ldots . + P_{k-2} z_1 + (P_{k-1} - y_1) = 0 \quad (254)$$

über und muss somit, da die Irreductibilität der Gleichung (252) angenommen war, eine identische sein. Daher wird auch jede Lösung der Gleichung (252) der Gleichung (254) genügen und sich daher für jede andere Lösung z_2 ebenfalls $z_2' = y_1$ oder

$$\int y_1 dx = z_2$$

ergeben, so dass

$$z_2 = z_1 + c$$

sein muss; da aber dies, wie oben nachgewiesen worden, für eine irreductible algebraische Gleichung nicht statthaben kann, so muss (252) vom ersten Grade also z_1 eine rationale Function von

$$x,\, y_1,\, y_1',\, \ldots .\, y_2,\, y_2',\, \ldots .\, u_1,\, \eta_1(u_1),\, \eta_1'(u_1),\, \ldots .$$

sein, und wir erhalten den Satz:

Ist die Quadratur eines Integrales einer algebraischen Differentialgleichung mit Hülfe der Integrale dieser Differentialgleichung und deren Ableitungen in algebraischer Form reducirbar auf das Integral einer algebraischen Differentialgleichung niederer Ordnung und dessen Ableitungen für ein Argument, das wieder algebraisch aus den Integralen der gegebenen Differentialgleichung und deren Ableitungen zusammengesetzt ist, so ist diese Reduction stets mit Adjungirung dieses letzten Elementes selbst in rationaler Form darstellbar, und dasselbe gilt offenbar, wenn in die Reductionsformel mehrere Integrale von algebraischen Differentialgleichungen niederer Ordnung, deren Argumente wiederum in der angegebenen Weise zusammengesetzt sind, eintreten.

Sei nun die rationale Reductionsformel

$$\int y_1 dx = \Omega(x, y_1, y_1', \ldots y_1^{(m)}, y_2, y_2', \ldots y_2^{(m)}, \ldots u_1, \eta_1(u_1), \eta_1'(u_1), \ldots \eta_1^{(\mu)}(u_1)), \tag{255}$$

so liefert das Differential dieser Gleichung

$$y_1 = \frac{\partial\Omega}{\partial x} + \frac{\partial\Omega}{\partial y_1} y_1' + \ldots + \frac{\partial\Omega}{\partial y_2} y_2' + \ldots + \frac{\partial\Omega}{\partial u_1} u_1' + \frac{\partial\Omega}{\partial \eta_1(u_1)} \eta_1'(u_1)\, u_1' + \ldots, \tag{256}$$

in welcher vermöge der Differentialgleichungen (247) und (250)

$$y_1^{(\mu+1)},\, y_2^{(\mu+1)},\, \ldots .\, \eta_1^{(\mu+1)}(u_1)$$

rational durch die niederen Ableitungen ausgedrückt ersetzt werden können.

Da in der Differentialgleichung (250) u_1 eine durch die Gleichung (251) definirte algebraische Function von $x, y_1, y_2, \ldots .$ und deren Ableitungen ist, so kann man derselben die oben bezeichneten Grössen adjungiren, un des mag die in Bezug auf den höchsten Differentialquotienten mit Adjungirung der Grössen $x, y_1, y_1', \ldots . y_1^{(m)}, y_2, y_2', \ldots . y_2^{(m)}, \ldots .$ und u_1 algebraisch irreductible Differentialgleichung niedrigster Ordnung, welche das Integral $\eta_1(u_1)$ besitzt,

$$\begin{aligned}\eta(u_1)^{(\rho)^\sigma} + \omega_1(x, u_1, y_1, y_1', \ldots y_2, y_2', \ldots, \eta(u_1), \eta'(u_1), \ldots \eta^{(\rho-1)}(u_1))\,\eta(u_1)^{(\rho)^{\sigma-1}} \\ + \ldots + \omega_\sigma(x, u_1, y_1, y_1', \ldots y_2, y_2', \ldots . \eta(u_1), \eta'(u_1), \ldots \eta^{(\rho-1)}(u_1)) = 0 \quad (257)\end{aligned}$$

sein. Da nun in der Gleichung (256) u_1' vermöge (251) als ganze Function von u_1 dargestellt werden kann, so kann man diese Gleichung kurz in der Form schreiben

$$\Omega_1(x, u_1, y_1, y_1', \ldots . y_2, y_2', \ldots \eta(u_1), \eta'(u), \ldots \eta^{(\mu)}(u_1)) = 0, \quad (258)$$

worin Ω_1 eine ganze Function bedeutet und welcher $\eta_1(u_1)$ Genüge leistet. Nach der für die Differentialgleichung (257) gemachten Annahme folgt aber aus den am Anfange der Arbeit durchgeführten Untersuchungen, dass jedes nicht singuläre Integral der Gleichung (257) auch (258) genügt, also auch (256) und somit durch Integration auch (255) befriedigt, dass also, wenn $\eta_2(u_1)$ irgend ein nicht singuläres Integral von (257) bedeutet,

$$\int y_1 dx = \Omega(x, y_1, y_1', \ldots y_1^{(m)}, y_2, y_2', \ldots y_2^{(m)}, \ldots u_1, \eta_2(u_1), \eta_2'(u_1), \ldots \eta_2^{(\mu)}(u_1)) \quad (259)$$

sein wird oder endlich durch Zusammenstellung mit (255)

$$\begin{aligned}&\Omega(x, y_1, y_1', \ldots y_1^{(m)}, y_2, y_2', \ldots y_2^{(m)}, \ldots u_1, \eta_2(u_1), \eta_2'(u), \ldots \eta_2^{(\mu)}(u_1)) \\ &= \Omega(x, y_1, y_1', \ldots y_1^{(m)}, y_2, y_2', \ldots y_2^{(m)}, \ldots u_1, \eta_1(u_1), \eta_1'(u_1), \ldots \eta_1^{(\mu)}(u_1)) + C, \quad (260)\end{aligned}$$

worin C eine Constante bedeutet. Diese Gleichung entspricht allgemein der oben für Differentialgleichungen erster Ordnung zu Grunde gelegten Functionalgleichung (208) u. f. und es ist vermöge der Eigenschaften der Differentialgleichung (257) sowohl die Möglichkeit als auch die Form der Gleichung (260) also der Function Ω, welche die Reduction des Abel'schen Integrales liefert-festzustellen.

Es ist bekannt, dass aus den von Abel aufgestellten Sätzen über die *rationale* Zurückführung von Integralen algebraischer Functionen auf algebraisch-logarithmische Functionen es gelungen ist, die nothwendigen und hinreichenden Bedingungen für die Möglichkeit einer solchen Zurückführung aufzustellen; wie sich die analogen Untersuchungen für Integrale transcendenter Functionen gestalten, soll den Gegenstand einer späteren Arbeit bilden.

HEIDELBERG den 28ten April 1888.

Note on the Double Periodicity of the Elliptic Functions.

BY F. FRANKLIN.

If a function w be defined by the equation

$$z=\int^{w}\frac{dw}{\sqrt{1-w^2}},$$

whence

$$dw=\sqrt{1-w^2}\,dz, \tag{1}$$

it follows geometrically from the definition that w is a singly periodic function of z, with a real period, viz. 2π. For if the complex variables z and w be represented by points in the usual way, equation (1), which may be written

$$dw=i\sqrt{w-1}\sqrt{w+1}\,dz,$$

shows that when z describes a straight line parallel to the axis of X, the tangent to the path of w makes with the axis of X an angle equal to $\frac{1}{2}\pi$ plus half the sum of the angles made with the axis of X by the lines drawn from w to 1 and -1; in other words, the normal to the path of w bisects internally the angle formed by the lines drawn from the point w to the points 1 and -1. Hence when z describes a straight line parallel to the axis of X, w describes an ellipse having the points 1 and -1 as foci; and therefore w returns periodically to a given value as z receives a certain increment ω. The value of ω is at once obtained, if, in equation (1), we attend to the moduli, instead of the arguments, of the factors. If we denote by s' the length of the path described by z, by s the corresponding length for w, and by r_1 and r_2 the focal radii of the point w, equation (1) gives

$$ds=\sqrt{r_1r_2}\,ds',$$

whence

$$s'=\int ds/\sqrt{r_1r_2}. \tag{2}$$

This integral, taken along the whole ellipse, gives the value of ω.

Likewise, when z moves parallel to the axis of Y, w describes a hyperbola having the points 1 and -1 as foci; and it is plain from (2) that z must describe an infinite distance in order that w shall go off to infinity on the hyperbola. Hence there is no pure imaginary period.

Since to two points w on the same hyperbola belong two values of z which differ by a pure imaginary, the period ω, which is real, must be the same for all the ellipses; but the limit of these ellipses is a circle of infinite radius, for which we have obviously, from (2),

$$\omega = \int_0^{2\pi} d\vartheta = 2\pi.$$

This value of the integral of $ds/\sqrt{r_1 r_2}$ taken along any ellipse, may also be obtained from the consideration that the mean proportional between the focal radii of a point is equal to the conjugate diameter; then the result follows by projection from the circle.

From the fact that an ellipse is cut in only two points by one branch of a confocal hyperbola, it follows that there cannot be more than two values of z, noncongruous with respect to the modulus 2π, belonging to a given value of w. Hence, *à fortiori*, there can be no other period.

If, to fix the ideas, we suppose the lower limit in the integral

$$z = \int^w dw/\sqrt{1-w^2}$$

to be 0, and write $w = \phi(z)$, we have $\phi(0) = 0$; and it is plain from the symmetry of the curves that $\phi(-z) = -\phi(z)$ and $\phi(\pi + z) = -\phi(z)$, so that $\phi(\pi - z) = \phi(z)$; thus the two values of z corresponding to a given value of w are z and $\pi - z$.

If, instead of the equation $dw = \sqrt{1-w^2}\, dz$, we take

$$dw = \sqrt{(a_1 - w)(a_2 - w)}\, dz,$$

where a_1 and a_2 are any complex quantities, it is plain that when z describes a straight line parallel to the axis of X, w describes an ellipse whose foci are at a_1 and a_2; and when z describes a straight line parallel to the axis of Y, w describes a confocal hyperbola. Hence, as before, w is a singly periodic function with the real period 2π. If $dw = \sqrt{(w - a_1)(w - a_2)}\, dz$, the ellipses are described when z moves parallel to the axis of X, the hyperbolas when z moves parallel to the axis of Y; so that w has the pure imaginary period $2\pi i$.

To go back to a still simpler case: the path of the function w defined by the equation $z = \int^w dw/w$, i. e.

$$dw = w dz$$

is obviously a straight line through the origin when z moves parallel to the axis of X, and a circle with its centre at the origin when z moves parallel to the axis of Y. Hence w is a singly periodic function with the period $2\pi i$.

The following theorem was given by Clifford without demonstration,* and has been proved by a special method by Crofton (London Math. Soc. Proc. (1867), II, 37): *If* 1, 2, 3, 4 *be four concyclic foci of a bicircular quartic through the point* P, *the tangent to the quartic at* P *bisects the angle between the circles* P_{12}, P_{34}. This property may be exhibited in a form which makes the double periodicity of the function defined as the inverse of an elliptic integral of the first kind follow geometrically from its definition, in the same way in which the single periodicity of the function inverse to $\int dw/\sqrt{(a_1 - w)(a_2 - w)}$ follows from the like property for the ellipse and hyperbola. Namely, if we denote by ϑ_{AB} the angle made with an arbitrary line of reference by the line joining any two points A and B, and by ϑ_t, $\vartheta_{t'}$, $\vartheta_{t''}$, the angles made with the line of reference by the tangents at P to the quartic and the two circles respectively, we have, by elementary geometry,

$$\vartheta_{t'} + \vartheta_{12} = \vartheta_{P1} + \vartheta_{P2}, \quad \vartheta_{t''} + \vartheta_{34} = \vartheta_{P3} + \vartheta_{P4};$$

but, by Clifford's theorem,

$$\vartheta_t = \frac{1}{2}(\vartheta_{t'} + \vartheta_{t''});$$

hence

$$\vartheta_t + \frac{1}{2}(\vartheta_{12} + \vartheta_{34}) = \frac{1}{2}(\vartheta_{P1} + \vartheta_{P2} + \vartheta_{P3} + \vartheta_{P4}).\dagger$$

If, now, we take the arbitrary line of reference parallel to a bisector of the angle formed by 12 and 34, $\vartheta_{12} + \vartheta_{34} = 0$, and the above equation becomes

$$\vartheta_t = \frac{1}{2}(\vartheta_{P1} + \vartheta_{P2} + \vartheta_{P3} + \vartheta_{P4});$$

that is, *the angle made with an* axis *of four concyclic foci of a bicircular quartic by the tangent to the quartic at any point is equal to half the sum of the angles made with the same axis by the four focal radii of the point;*‡ an *axis* of four concyclic points being understood to be any line parallel to a bisector of the angle formed by any pair of opposite sides of the quadrangle formed by the four points; the directions of the bisectors are of course the same for every pair.

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* Proposed as a problem in the Educational Times, March 1866, and again January 1874.

† Of course, in these equations, quantities which differ by a multiple of π are considered equal.

‡ A simple algebraic proof of this theorem will be given in a subsequent article in this Journal, on bicircular quartics.

If the four real foci 1, 2, 3, 4 of a real bicircular quartic are not concyclic, they are such that ($I3$, $J4$) and ($I4$, $J3$) are foci concyclic with 1 and 2; or, in other words, two of the foci are concyclic with the antipoints of the other two. It is easily seen that the sum of the angles made with an arbitrary line by the lines joining a given point to a pair of points is not altered by the substitution of the antipoints for the original pair of points. Also, since the line joining the antipoints is perpendicular to the line joining the original pair, the "axes" of the concyclic foci are the bisectors of the angles between 12 and a perpendicular to 34. Hence if, in the case of four points such that the antipoints of one pair are concyclic with the other pair (or, what is the same thing, such that either pair are conjugate points on a diameter of a circle through the other pair; which again is equivalent to the statement that $13:23=14:24$) we define an axis of the four points to be a line parallel to a bisector of the angle formed by the connector of one pair and a perpendicular to the connector of the other pair, the above theorem holds for the four *real* foci, whether they are concyclic or not.

Consider, now, the function defined by the equation

$$dw=\sqrt{(w-a_1)(w-a_2)(w-a_3)(w-a_4)}\,dz.$$

When z describes a straight line, w describes a curve whose tangent makes with the path of z an angle equal to half the sum of the angles made with the axis of X by the lines drawn from w to the points a_1, a_2, a_3, a_4; or, say, the points 1, 2, 3, 4. If, then, these points are concyclic, or if they are such that $13:23=14:24$; and if the axes of the four points are parallel to the axes of X and Y; then, when z describes a line *parallel to either axis*, w describes one of the bicircular quartics having the four points as foci;* these quartics forming two systems orthogonal to each other. But a bicircular quartic is a closed curve not in general passing through any focus; hence w is a doubly periodic function with a real period ω and a pure imaginary period $i\omega'$. The value of ω is

$$\int ds/\sqrt{r_1r_2r_3r_4}$$

---

*I find that Siebeck (Crelle (1860), 57, 359) showed that sn $(u+iv)$ describes a bicircular quartic when u or v is constant; and that Greenhill (Camb. Philos. Soc. Proc. (1881), 4, 77) showed the like to be true of the function inverse to $\int dw/\sqrt{(w-a_1)(w-a_2)(w-a_3)(w-a_4)}$, if the a's are concyclic. The point of view of these papers is, however, entirely different from that of the present note.

taken along a complete oval belonging to one system of the quartics; ω' is the value of the same integral taken along an oval belonging to the opposite system. And we have, incidentally, the theorem that this integral has a constant value for all ovals belonging to the same system of confocal bicircular quartics. As a limiting case, the integral of $ds/\sqrt{r_1r_2r_3r_4}$, taken along the oval or the infinite branch of the circular cubic belonging to one system has the value ω; for the circular cubic belonging to the opposite system the value is ω'.

It may be noted that there is a definite distinction between the two systems of quartics; viz. if it is understood that the focal radii are drawn in a definite sense, either all *from* or all *to* the foci, then, in the quartics which correspond to the real period, the *tangent* makes with the axis of X an angle equal to half the sum of the angles made with it by these focal radii; and in the quartics which correspond to the imaginary period, the *normal* does so.

Of course, if we have

$$dw = \sqrt{(a_1 - w)(a_2 - w)(a_3 - w)(a_4 - w)}\, dz,$$

the only difference is that the two systems of quartics change places.

More generally, let

$$dw = ae^{i\lambda} \sqrt{(w - a_1)(w - a_2)(w - a_3)(w - a_4)}\, dz,$$

the points 1, 2, 3, 4 being either concyclic or such that $13:23 = 14:24$, and let the axes of the four points make the angles μ and $\mu + \frac{1}{2}\pi$ with the axis of X; then it is plain that when z moves along a line making an angle $-(\lambda + \mu)$ or $-(\lambda + \mu + \frac{1}{2}\pi)$ with the axis of X, w describes a bicircular quartic having the points 1, 2, 3, 4 as foci. Hence w has two periods, whose ratio is a pure imaginary; and the condition that one period be real and the other pure imaginary is that the axes of the four points 1, 2, 3, 4 make an angle $-\lambda$ with the axes of X and Y respectively.

The function inverse to

$$\int dw/\sqrt{a + bw + cw^2 + dw^3 + ew^4}$$

always comes under the case we have been considering *if the coefficients are real.* Namely, when the roots are real, the points 1, 2, 3, 4 are collinear and therefore concyclic; when 1 and 2 are real and 3 and 4 are conjugate imaginaries, the antipoints of 3 and 4 are collinear with 1 and 2; and when the roots are two pairs of conjugate imaginaries, the points are evidently concyclic. In all these cases, the axes of the points are evidently parallel to the axes of X and Y;

consequently the periods are one of them real and the other pure imaginary. In all these cases, too, it may be observed, the quartics have four collinear foci, the line of them being the axis of X; this has been mentioned in the first two cases, and it is true of the third case because the antipoints of the pair of points representing a pair of conjugate imaginaries lie on the axis of X.

To bring the case of

$$dw = ae^{i\lambda}\sqrt{(w-\alpha_1)(w-\alpha_2)(w-\alpha_3)}\,dz$$

under the foregoing, we must regard α_4 as having gone to infinity *along the axis of* X, so that the angle made with the axis of X by the line joining w to α_4 shall be 0, and thus not affect the construction for the tangent. In order that the four points be concyclic, 123 must be collinear; in order that the antipoints of 1 and 2 be concyclic with 3 and 4, 3 must be equidistant from 1 and 2. The paths corresponding to the periods are in this case Cartesians whose real foci are at 1, 2, 3 (the focus at infinity being ignored).

In particular,

$$dw = \sqrt{a + bw + cw^2 + dw^3}\,dz$$

falls under these heads if the coefficients are real. In this case, the three real foci are in the axis of X when the roots are real; one real focus and two imaginary foci are in the axis of X when two of the roots are imaginary; and in either event the axes of the root-points are parallel to the axes of X and Y, so that there is a real period and a pure imaginary period.

There is nothing to prevent our dropping one more factor, and considering the path corresponding to

$$dw = \sqrt{(w-\alpha_1)(w-\alpha_2)}\,dz$$

as arising from the preceding by α_3 as well as α_4 going to infinity along the axis of X. Only, in this case, one of the periods, which is $\int ds/\sqrt{r_1 r_2}$ taken along a hyperbola, becomes infinite, the denominator being an infinite of the first order.

If a function w be defined by the equation

$$z = \int^{w} f(w)\,dw, \tag{1}$$

$f(w)$ being *any* function with real coefficients, then the general equation of the curves described by w when z moves parallel to the axis of Y furnishes the addition-equation of the function w. For if we write

$$w = \phi(z),\ z = \phi^{-1}(w), \tag{2}$$

then, since the coefficients in $f(w)$ are real, there is evidently no loss of generality in assuming

$$\phi^{-1}(X+iY) = \text{conjugate of } \phi^{-1}(X-iY); \tag{3}$$

hence (denoting the rectangular coordinates of z by X and Y) to say that z describes a line parallel to the axis of Y is the same as to say that

$$\phi^{-1}(X+iY) + \phi^{-1}(X-iY) = \text{const.} \tag{4}$$

Therefore the equation of the curves described by w when z describes any line parallel to the axis of Y is equivalent to equation (4). Let us, then, put

$$X+iY=x,\ X-iY=y; \tag{5}$$

equation (4) becomes

$$\phi^{-1}(x) + \phi^{-1}(y) = \text{const.} \tag{6}$$

and the equation of the w-curves becomes an equation of the form

$$F(x, y, C) = 0, \tag{7}$$

which, being equivalent to (6), is the addition-equation of the function ϕ.

Of course the connection between (6) and (7), or, say, between the differential equation $$f(x)\,dx + f(y)\,dy = 0$$
and the equation $$F(x, y, C) = 0$$

is a purely analytic one. Having once obtained the latter equation, we have no longer anything to do with the original significance of x and y, nor with the question of the reality of the coefficients in f.

What we have found, then, may be stated as follows: Let a function w be defined by the equation $$z = \int^w f(w)\,dw,$$

the coefficients in f being literal. If, on the supposition that these coefficients are real, the general equation of the curves described by w when z moves parallel to the axis of Y, converted into an equation in x and y by the substitution

$$x = X+iY,\ y = X-iY,$$

is $$F(x, y, C) = 0,$$

then this equation is the solution of the equation

$$f(x)\,dx + f(y)\,dy = 0,$$

or, in other words, is the addition-equation of the function w; and the result holds, of course, whether the coefficients in f be real or imaginary.

In like manner, the equation of the curves described by w when z moves parallel to the axis of X, converted into an equation in x and y by the substitution
$$x = X + iY,\ y = X - iY,$$
is the solution of the equation
$$f(x)\,dx - f(y)\,dy = 0\,.$$
If the two sets of curves are comprised in a single equation, this is, of course, the solution of
$$f(x)\,dx \pm f(y)\,dy = 0\,.$$

For example, to find the addition-equation of the function w defined by the equation
$$z = \int^{w} dw / \sqrt{Aw^2 + 2Bw + C}\,.$$
We know that when z moves parallel to the axes of X and Y, the paths of w are the confocal ellipses and hyperbolas having their foci at the root-points of the quadratic under the radical sign. Expressing everything in the circular coordinates x and y $(= X + iY,\ X - iY)$ the foci are evidently given, *when* A, B, C *are real*, by the equations
$$Ax^2 + 2Bxz + Cz^2 = 0\,,\ Ay^2 + 2Byz + Cz^2 = 0\,,$$
the z being introduced for homogeneity. Hence the equation of the required curves is
$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0\,,$$
with such conditions among the coefficients as will make the preceding pair of equations represent the pairs of tangents from (x, z) and (y, z) respectively. These conditions are evidently
$$c_1 : g_1 : a_1 = A : -B : C,\quad c_1 : f_1 : b_1 = A : -B : C,$$
where a_1 is the minor of a in the determinant of the conic, etc. Hence we at once obtain
$$\begin{array}{ccccccccccc} a & : & b & : & c & : & f & : & g & : & h \end{array}$$
$$= AC - B^2 : AC - B^2 : C^2 - H^2 : BC - BH : BC - BH : B^2 - AH,$$
H being an arbitrary constant. Therefore the equation of our required system of curves, or in other words, the solution of the differential equation
$$\frac{dx}{\sqrt{Ax^2 + 2Bx + C}} \pm \frac{dy}{\sqrt{Ay^2 + 2By + C}} = 0$$
is
$$(AC - B^2)(x^2 + y^2) + 2B^2xy + 2BC(x + y) + C^2 - 2H\{Axy + B(x + y)\} - H^2 = 0,$$

the result holding true, of course, whether A, B, C are real or not. The result might, of course, have been otherwise obtained by writing the equation of the confocal conics in X and Y and then transforming to x and y. In particular, the integral of

$$\frac{dx}{\sqrt{1-x^2}} \pm \frac{dy}{\sqrt{1-y^2}} = 0,$$

obtained from the above by putting $A, B, C = -1, 0, 1$, is

$$x^2 + y^2 - 1 - 2Hxy + H^2 = 0,$$

whence

$$H = xy \pm \sqrt{(1-x^2)(1-y^2)},$$

which is equivalent to the formula

$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v.$$

In the same way, to obtain the addition-equation of the function w defined by the equation

$$z = \int^w dw / \sqrt{Aw^4 + 4Bw^3 + 6Cw^2 + 4Dw + E},$$

in which the coefficients are perfectly general, we have only to find the equation, in circular coordinates, of the bicircular quartics whose foci are given by

$$Ax^4 + 4Bx^3z + 6Cx^2z^2 + 4Dxz^3 + Ez^4 = 0, \quad Ay^4 + 4By^3z + 6Cy^2z^2 + 4Dyz^3 + Ez^4 = 0;$$

that is, of a system of confocal bicircular quartics, four of whose foci (not necessarily the real ones) are in a straight line. The solution of this problem (which presents no difficulty) will be contained in an article shortly to appear in this Journal, on bicircular quartics.

But it is to be observed that while the method we have been illustrating gives the *addition-equation* of the function w for the most general case, it gives the *path* of w (corresponding to a motion of z parallel to the axes) only in the special case when the coefficients are real. So far as the elliptic functions are concerned, we had found the paths of w corresponding to two periods in the more general case when the root-points of the biquadratic are concyclic or what may be called anticoncyclic; but we have not considered the path of w at all in the general case where no restriction is placed upon the position of the root-points. What we want in this case is a curve such that the tangent at any point makes with some axis an angle equal to half the sum of the angles made with the axis by the lines joining the point to four fixed points whose position is arbitrary. I have not succeeded in geometrically determining such a curve.*

Finally, it may be worth while to make an obvious remark concerning the cases which have been considered. Since the sum of any integer multiples of two periods is a period, and conversely, the path of w will be a closed curve not only when z moves in the two mutually perpendicular directions corresponding to the periods already noticed, but also whenever z describes a straight line, the tangent of whose angle with either of these directions is the ratio of any integer multiples of ω and ω'; and in no other case. Hence we have the geometrical theorem that the oblique trajectory of a system of confocal bicircular quartics, the angle of intersection being α, is a closed curve for an infinite number of values of α; viz. whenever

$$\tan\alpha = \frac{m\int_S ds/\sqrt{r_1 r_2 r_3 r_4}}{n\int_{S'} ds/\sqrt{r_1 r_2 r_3 r_4}},$$

m and n being integers, and $\int_S$ and $\int_{S'}$ being taken along complete ovals belonging to opposite systems.

In the case of the confocal conics, the corresponding function having only one period, none of the oblique trajectories are closed curves.

---

* Its differential equation in circular coordinates is, however, obvious, and is immediately integrable by elliptic functions. See an article on "Some Applications of Circular Coordinates," to appear shortly in this Journal.

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CONTENTS.

PAGE

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Die Herstellung einer linearen Differentialgleichung aus einem gegebenen Element der Integralfunction. Von **KARL HEUN**, ... 215

Über die Reduction von Integralen transcendenter Functionen. Von **LEO KOENIGSBERGER**, ... 221

Note on the Double Periodicity of the Elliptic Functions. By **F. FRANKLIN**, . . 283

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On the Surfaces with Plane or Spherical Curves of Curvature.

(For first part see vol. XI, No. 1.)

By Prof. Cayley.

I consider next the case

$$PS4^0 = \text{Serret's third case of } PS.$$

The six equations are

$$\begin{aligned} A^2 + B^2 + C^2 &= 1, \\ ax + by &= lm, \\ Aa + Bb &= l, \\ x^2 + y^2 + z^2 - 2\theta z &= 2v, \\ Ax + By + C(z - \theta) &= m, \\ Adx + Bdy + Cdz &= 0; \end{aligned}$$

where θ has been written in place of γ: m is a given constant; a, b, l are functions of t; v is a function of θ. The equation $ax + by = lm$ evidently denotes that the planes of the plane curves of curvature are all of them parallel to the axis of z, or what is the same thing, they envelope a cylinder; in the particular case $m = 0$, they all of them pass through the axis of z. In the general case, the required surface is the parallel surface, at the normal distance m, to the surface which belongs to the particular case $m = 0$. This is not assumed in the investigation which follows, but it will be readily perceived how the theorem is involved in, and in fact proved by, the investigation.

I obtain the solution synthetically as follows:

Taking T, a, b functions of t, $\mathrm{a}^2 + \mathrm{b}^2 = 1$; T_1, a_1, b_1 their derived functions, $\mathrm{a}\mathrm{a}_1 + \mathrm{b}\mathrm{b}_1 = 0$; $\Omega = \dfrac{T_1^2}{4T^2} + \mathrm{a}_1^2 + \mathrm{b}_1^2$; Θ a function of θ, Θ' its derived function,

$M = \frac{2\Theta}{\Theta'}$; $P = \frac{2\sqrt{T\Theta}}{T+\Theta}$, $Q = \frac{T-\Theta}{T+\Theta}$, and therefore $P^2 + Q^2 = 1$; then writing

$$\begin{aligned} A_0 &= \frac{1}{\sqrt{\Omega}}\Big(-\mathrm{b}\frac{T_1}{2T} + \mathrm{b}_1 Q\Big), \\ B_0 &= \frac{1}{\sqrt{\Omega}}\Big(\ \ \mathrm{a}\frac{T_1}{2T} - \mathrm{a}Q\Big), \\ C_0 &= \frac{-1}{\sqrt{\Omega}}(\mathrm{ab}_1 - \mathrm{a}_1\mathrm{b})\,P, \end{aligned}$$

(where $A_0^2 + B_0^2 + C_0^2 = 1$) we assume

$$\begin{aligned} x &= mA_0 \qquad + \mathrm{a}MP, \\ y &= mB_0 \qquad + \mathrm{b}MP, \\ z &= mC_0 + \theta + \ \ MQ, \end{aligned}$$

equations which determine x, y, z as functions of the parameters t and θ. As will presently be shown, we have $A_0dx + B_0dy + C_0dz = 0$; and we have thus $A, B, C = A_0, B_0, C_0$; and this being so, we easily verify the six equations

$$\begin{aligned} A^2 + B^2 + C^2 \qquad &= 1, \\ \mathrm{b}x - \mathrm{a}y \qquad &= m\Big(-\frac{T_1}{2T}\,\frac{1}{\sqrt{\Omega}}\Big), \\ \mathrm{b}A - \mathrm{a}B \qquad &= \ \ \Big(-\frac{T_1}{2T}\,\frac{1}{\sqrt{\Omega}}\Big), \\ x^2 + y^2 + (z-\theta)^2 &= m^2 + M^2 + \theta^2, \\ Ax + By + C(z-\theta) &= m, \\ Adx + Bdy + Cdz \ \ &= 0, \end{aligned}$$

which are the six equations of the problem with the values $a = \mathrm{b}$, $b = -\mathrm{a}$, $l = -\frac{2T_1}{T}\frac{1}{\sqrt{\Omega}}$, $2v = m^2 + M^2$, for a, b, l and $2v$.

We in fact at once obtain the third equation $\mathrm{b}A_0 - \mathrm{a}B_0 = -\frac{T_1}{2T}\frac{1}{\sqrt{\Omega}}$, and thence the second equation $\mathrm{b}x - \mathrm{a}y = m(\mathrm{b}A_0 - \mathrm{a}B_0)$, $= m\Big(-\frac{T_1}{2T}\frac{1}{\sqrt{\Omega}}\Big)$; then for the fifth equation we have

$$A_0x + B_0y + C_0(z-\theta) = m + M\{(A_0\mathrm{a} + B_0\mathrm{b})\,P + C_0Q\}, = m,$$

since $(A_0\mathrm{a} + B_0\mathrm{b})P + C_0Q = 0$; and for the fourth equation we have

$$x^2 + y^2 + (z-\theta)^2 = m^2 + 2m\{M(A_0\mathrm{a} + B_0\mathrm{b})\,P + CQ\} + M^2, = m^2 + M^2.$$

It remains only to prove the assumed equation $A_0dx + B_0dy + C_0dz = 0$. Writing for a moment $X, Y, Z = \mathrm{a}MP, \mathrm{b}MP, \theta + MQ$ we have

$$\begin{aligned} A_0dx + B_0dy + C_0dz &= A_0(mdA_0 + dX) + B_0(mdB_0 + dY) + C_0(mdC_0 + dZ), \\ &= A_0dX + B_0dY + C_0dZ, \end{aligned}$$

since $A_0dA_0 + B_0dB_0 + C_0dC_0 = 0$ in virtue of $A_0^2 + B_0^2 + C_0^2 = 1$.

We have thus to show that if $X, Y, Z = \mathrm{a}MP, \mathrm{b}MP, \theta + MQ$, then $A_0dX + B_0dY + C_0dZ = 0$; say we have

$$\begin{aligned} dX &= pdt + p'd\theta, \\ dY &= qdt + q'd\theta, \\ dZ &= rdt + r'd\theta, \end{aligned}$$

then the required values of A_0, B_0, C_0 are proportional to $qr' - q'r$, $rp' - r'p$, $pq' - p'q$, and the sum of their squares is $= 1$. Writing for shortness $MP = R$, $MQ = S$, we have

$$\begin{aligned} p &= \mathrm{a}_1R + \mathrm{a}R_1, & p' &= \mathrm{a}R', \\ q &= \mathrm{b}_1R + \mathrm{b}R_1, & q' &= \mathrm{b}R', \\ r &= \phantom{\mathrm{b}_1R + \mathrm{b}}S_1, & r' &= 1 + S'; \end{aligned}$$

hence

$$\begin{aligned} qr' - q'r &= \mathrm{b}\,[R_1(1 + S') - R'S_1] + \mathrm{b}_1R(1 + S'), \\ rp' - r'p &= -\mathrm{a}\,[R_1(1 + S') - R'S_1] - \mathrm{a}_1R(1 + S'), \\ pq' - p'q &= -\ (\mathrm{a}\mathrm{b}_1 - \mathrm{a}_1\mathrm{b})\,RR'. \end{aligned}$$

Here

$$\begin{aligned} R' &= MP' + M'P, & R_1 &= MP_1, \\ S' &= MQ' + M'Q, & S_1 &= MQ_1, \end{aligned}$$

and hence

$$\begin{aligned} RS' - R'S &= M^2(PQ' - P'Q), \\ R_1S' - R'S_1 &= M^2(P_1Q' - P'Q_1) + MM'(P_1Q - PQ_1); \end{aligned}$$

moreover, from the values of P and Q we have

$$\begin{aligned} P' &= \frac{\Theta'}{2\Theta}PQ, = \frac{PQ}{M}, & P_1 &= -\frac{T_1}{2T}PQ, \\ Q' &= -\frac{\Theta'}{2\Theta}P^2 = -\frac{P^2}{M}, & Q_1 &= \frac{T_1}{2T}P^2, \end{aligned}$$

and thence

$$P_1Q - PQ_1 = -\frac{T_1P}{2T}; \quad PQ' - P'Q = -\frac{P}{M}; \quad P_1Q' - P'Q_1 = 0;$$

also

$$\begin{array}{lll} R' = PQ + M'P, = P(Q+M'), & & 1+S = 1 - P^2 + M'Q, = Q(Q+M'); \\ R_1 = -\dfrac{T_1}{2T} MPQ\ , & RS' - R'S = -PM, & R_1S' - R'S_1 = -\dfrac{T_1}{2T} MPM', \end{array}$$

and consequently

$$\begin{aligned} R_1(1+S') - R'S_1 &= -\frac{T_1}{2T} MP\ (Q+M'), \\ R\,(1+S') &= MPQ(Q+M'), \\ RR' &= MP^2\ (Q+M'), \end{aligned}$$

and hence the foregoing expressions for $qr' - q'r$, $rp' - r'p$, $pq' - p'q$ each contain the factor $MP(Q+M')$; and, omitting this factor, the expressions are

$$\left\{-\mathrm{b}\frac{T_1}{2T} + \mathrm{b}_1 Q\right\},\ \left\{\mathrm{a}\frac{T_1}{2T} - \mathrm{a}_1 Q\right\},\ -(\mathrm{ab}_1 - \mathrm{a}_1\mathrm{b})\,P;$$

the sum of the squares of these values is $= \dfrac{T_1^2}{4T^2} + \mathrm{a}_1^2 + \mathrm{b}_1^2, = \Omega$, and we have thus the required values

$$\begin{aligned} A_0 &= \frac{1}{\sqrt{\Omega}}\left\{-\mathrm{b}\frac{T_1}{2T} + \mathrm{b}_1 Q\right\}, \\ B_0 &= \frac{1}{\sqrt{\Omega}}\left\{\quad \mathrm{a}\frac{T_1}{2T} - \mathrm{a}_1 Q\right\}, \\ C_0 &= \frac{-1}{\sqrt{\Omega}}(\mathrm{ab}_1 - \mathrm{a}_1\mathrm{b})\,P, \end{aligned}$$

which completes the proof.

In the case $m = 0$, the solution is

$$x,\ y,\ z = \frac{2\Theta}{\Theta'}\,\mathrm{a}\,\frac{2\sqrt{T\Theta}}{T+\Theta},\ \ \frac{2\Theta}{\Theta'}\,\mathrm{b}\,\frac{2\sqrt{T\Theta}}{T+\Theta},\ \ \theta + \frac{2\Theta}{\Theta'}\ \frac{T-\Theta}{T+\Theta}.$$

Bonnet, in the paper (Jour. Ecole Polyt. t. 20) referred to at the beginning of this memoir, gives for this case (see p. 199) a solution which he says is equivalent to that obtained by Joachimsthal in the paper "Demonstrationes theorematum ad superficies curvas spectantium," Crelle, t. 30 (1846), pp. 347–350; viz. Joachimsthal's form is

$$\begin{aligned} x &= \frac{\mu \sin L \sin \lambda}{1 + \cos L \cos M}, \\ y &= \frac{\mu \sin L \cos \lambda}{1 + \cos L \cos M}, \\ z &= \frac{\mu \cos L \sin M}{1 + \cos L \cos M} + \int \cot M\, d\mu, \end{aligned}$$

where L, M denote arbitrary functions of the parameters λ, μ respectively. To identify these with the foregoing form, I write

$$\sin\lambda = -\mathrm{a},\ \cos L = -\frac{T-1}{T+1},\ \cos M = \frac{\Theta-1}{\Theta+1}\quad,\ \mu = \frac{4\Theta\sqrt{\Theta}}{\Theta'(\Theta+1)};$$

$$\cos\lambda = -\mathrm{b},\ \sin L = \frac{-2\sqrt{T}}{T+1}\ ,\ \sin M = \frac{-2\sqrt{\Theta}}{\Theta+1};$$

we thus have

$$\frac{\sin L\sin\lambda}{1+\cos L\cos M} = \mathrm{a}\,\frac{2\sqrt{T}}{T+1} \div 1 - \frac{(T-1)(\Theta-1)}{(T+1)(\Theta+1)},\ = \frac{\mathrm{a}\sqrt{T}(\Theta+1)}{T+\Theta},$$

and thence

$$x = \frac{2\Theta}{\Theta'}\,\mathrm{a}\,\frac{2\sqrt{T\Theta}}{T+\Theta},\ y = \frac{2\Theta}{\Theta'}\,\mathrm{b}\,\frac{2\sqrt{T\Theta}}{T+\Theta}.$$

Moreover,

$$\frac{\cos L\sin M}{1+\cos L\cos M} = \frac{\sqrt{\Theta}\,(T-1)}{T+\Theta},$$

and thence the first term of z is $= \dfrac{2\Theta}{\Theta'}\,\dfrac{2\Theta\,(T-1)}{(\Theta+1)(T+\Theta)}$; or observing that $2\Theta\,(T-1) = (\Theta+1)(T-\Theta) + (\Theta-1)(T+\Theta)$, this is

$$= \frac{2\Theta}{\Theta'}\,\frac{T-\Theta}{T+\Theta} + \frac{2\Theta\,(\Theta-1)}{\Theta'(\Theta+1)},$$

or we have

$$z = \frac{2\Theta}{\Theta'}\,\frac{T-\Theta}{T+\Theta} + \frac{2\Theta\,(\Theta-1)}{\Theta'(\Theta+1)} + \int\cot M\,d\mu.$$

Here

$$\begin{aligned}\cot M\,d\mu &= -\frac{\Theta-1}{2\sqrt{\Theta}}\,\frac{4\Theta\sqrt{\Theta}}{\Theta(\Theta+1)}\left\{\frac{\tfrac{3}{2}\Theta'}{\Theta} - \frac{\Theta'}{\Theta+1} - \frac{\Theta''}{\Theta'}\right\}d\theta\\ &= \frac{\Theta\,(\Theta-1)}{\Theta+1}\left\{-\frac{3}{\Theta} + \frac{2}{\Theta+1} + \frac{2\Theta''}{\Theta'^2}\right\}d\theta.\end{aligned}$$

But writing

$$\xi = \frac{2\Theta\,(\Theta-1)}{\Theta'(\Theta+1)},$$

we have

$$\begin{aligned}d\xi &= \frac{2\Theta\,(\Theta-1)}{\Theta'(\Theta+1)}\left\{\frac{\Theta'}{\Theta} + \frac{\Theta'}{\Theta-1} - \frac{\Theta'}{\Theta+1} - \frac{\Theta''}{\Theta'}\right\}d\theta\\ &= \frac{\Theta\,(\Theta-1)}{\Theta+1}\left\{\frac{2}{\Theta} + \frac{2}{\Theta-1} - \frac{2}{\Theta+1} - \frac{2\Theta''}{\Theta'^2}\right\}d\theta,\end{aligned}$$

and thence

$$d\xi + \cot M\,d\mu = \frac{\Theta(\Theta-1)}{\Theta+1}\left\{-\frac{1}{\Theta}+\frac{2}{\Theta-1}\right\}d\theta, = d\theta,.$$

and consequently

$$\xi + \int \cot M\,d\mu = \theta,$$

and the value of z thus is

$$z = \theta + \frac{2\Theta}{\Theta'}\,\frac{T-\Theta}{T+\Theta},$$

which completes the identification.

Bonnet's formulae just referred to, making a slight change of notation and correcting a sign, are

$$x = \frac{\Gamma'\sin\theta}{\cos i(c+\Theta)},$$

$$y = \frac{\Gamma'\cos\theta}{\cos i(c+\Theta)},$$

$$z = \Gamma + i\Gamma'\tan i(c+\Theta),$$

where Γ, Θ are arbitrary functions of the parameters c, θ respectively. To identify these with Joachimsthal's, write

$$\begin{array}{llll} \sin\lambda = \sin\theta, & \cos M = \;i\cot ic, & \cos L = \;i\cot i\Theta, & \mu = -\Gamma'\operatorname{cosec} ic, \\ \cos\lambda = \cos\theta, & \sin M = \operatorname{cosec} ic, & \sin L = \operatorname{cosec} i\Theta, & \\ & \cot M = \;i\cos ic, & \cot L = \;i\cos i\Theta, & \end{array}$$

we have

$$x = \frac{\mu\operatorname{cosec} i\Theta\sin\theta}{1-\cot ic\cot i\Theta} = \frac{-\mu\sin ic\sin\theta}{\cos i(c+\Theta)}, = \frac{\Gamma'\sin\theta}{\cos i(c+\Theta)};$$

and similarly

$$y = \qquad\qquad\qquad\qquad \frac{\Gamma'\cos\theta}{\cos i(c+\Theta)}.$$

Moreover, the first term of z is

$$\frac{\mu i\cot i\Theta\operatorname{cosec} ic}{1-\cot ic\cot i\Theta} = \frac{-\mu i\cos i\Theta}{\cos i(c+\Theta)} = \frac{i\Gamma'\cos i\Theta}{\sin ic\cos i(c+\Theta)};$$

or since $i\Theta = i(c+\Theta) - ic$, and thence

$$\cos i\Theta = \cos ic\cos i(c+\Theta) + \sin ic\sin i(c+\Theta),$$

this is

$$= i\Gamma'\{\cot ic + \tan i(c+\Theta)\},$$

and we have

$$z = i\Gamma'\tan i(c+\Theta) + i\Gamma'\cot ic + \int\cot M\,d\mu.$$

But from the equation $\mu = -\Gamma' \operatorname{cosec} ic$, we obtain

$$d\mu = (-\Gamma'' \operatorname{cosec} ic + i\Gamma' \operatorname{cosec} ic \cot ic)\, dc;$$

whence

$$\cot M d\mu = (-i\Gamma'' \cot ic - \Gamma' \cot^2 ic)\, dc,$$

and thence

$$\begin{aligned} d\left(i\Gamma' \cot ic + \int \cot M\, d\mu\right) &= (i\Gamma'' \cot ic + \Gamma' \operatorname{cosec}^2 ic)\, dc \\ &\quad + (-i\Gamma'' \cot ic - \Gamma' \cot^2 ic)\, dc, = \Gamma' dc; \end{aligned}$$

that is

$$i\Gamma' \cot ic + \int \cot M\, d\mu = \Gamma,$$

and consequently

$$z = \Gamma + i\Gamma' \tan i(c + \Theta),$$

which completes the identification of Bonnet's formula with Joachimsthal's.

SS. THE SETS OF CURVES OF CURVES OF CURVATURE EACH SPHERICAL.

The six equations are

$$\begin{aligned} A^2 + B^2 + C^2 &= 1, \\ x^2 + y^2 + z^2 - 2ax - 2by - 2cz - 2u &= 0, \\ A(x-a) + B(y-b) + C(z-c) - l &= 0, \\ x^2 + y^2 + z^2 - 2\alpha x - 2\beta y - 2\gamma z - 2v &= 0, \\ A(x-\alpha) + B(y-\beta) + C(z-\gamma) - \lambda &= 0, \\ Adx + Bdy + Cdz &= 0; \end{aligned}$$

the condition being

$$a\alpha + b\beta + c\gamma - l\lambda + u + v = 0.$$

The cases are

	a	b	c	l	u	α	β	γ	λ	v
*SS*1°	0	0	0	l	$\tfrac{1}{2}(ml + m')$	α	β	γ	$\tfrac{1}{2} m$	$-\tfrac{1}{2} m'$
*SS*2°	0	0	c	l	$\tfrac{1}{2}(ml + m')$	α	β	0	$\tfrac{1}{2} m$	$-\tfrac{1}{2} m'$
*SS*3°	0	0	c	$mc + \tfrac{1}{2} m'$	$-\tfrac{1}{2} m''c - m'''$	α	β	$m\lambda + \tfrac{1}{2} m''$	λ	$\tfrac{1}{2} m'\lambda + m'''$
*SS*4°	0	b	c	$mc + m'$	$mm''c + m'm'' - m'''$	α	0	γ	$\frac{1}{m}\gamma + m''$	$\frac{m'}{m}\gamma + m'''$

where m, m', m'', m''' are constants; b, c, l functions of t; $\alpha, \beta, \gamma, \lambda$ functions of θ.

*SS*1° gives circles (i. e. the curves of curvature of one set are circles).

*SS*2° is Serret's first case of *SS*.

*SS*3° gives circles.

*SS*4° is Serret's second case of *SS*.

$SS2^0$ = SERRET'S FIRST CASE OF SS.

Writing for convenience $m' = -f^2$, the six equations are

$$\begin{aligned}
&A^2+B^2+C^2 &&=1,\\
&x^2+y^2+z^2-2cz-ml+f^2 &&=0,\\
&Ax+By+C(z-c)-l &&=0,\\
&x^2+y^2+z^2-2\alpha x-2\beta y-f^2 &&=0,\\
&A(x-\alpha)+B(y-\beta)+C(z-\lambda) &&=0,\\
&Adx+Bdy+Cdz &&=0,
\end{aligned}$$

where m, f are constants; c, l are functions of t; α, β, λ functions of θ. The first set of spheres have no points in common, but the second set have in common the two points $x=0$, $y=0$, $z=\pm f$. Hence inverting (by reciprocal radius vectors) with one of these points, say $(0, 0, f)$ as centre, the spheres of the first set will continue spheres, but the spheres of the second set will be changed into planes, and the required surface is thus the inversion of a surface PS, which is in fact $PS3^0$: say this surface PS is the "Inversion" of SS. We invert by the formulae

$$x=\frac{K^2X}{\Omega},\quad y=\frac{K^2Y}{\Omega},\quad z-f=\frac{K^2(Z-f)}{\Omega},$$

where $\Omega = X^2+Y^2+(Z-f)^2$.

Writing the equation for the second set of spheres in the form

$$x^2+y^2+(z-f)^2-2\alpha x-2\beta y+2f(z-f)=0,$$

the transformed equation is at once found to be

$$-2\alpha X-2\beta Y+2f(Z-f)+K^2=0,$$

or say
$$\alpha X+\beta Y-fZ+f^2-\tfrac{1}{2}K^2=0;$$

viz. this gives the planes of the Inversion.

Similarly for the first set of spheres, writing the equation in the form

$$x^2+y^2+(z-f)^2+2(f-c)(z-f)+2f(f-c)-ml=0,$$

the transformed equation is found to be

$$\{2f(f-c)-ml\}\{X^2+Y^2+(Z-f)^2\}+2(f-c)K^2(Z-f)+K^4=0;$$

viz. this is

$$\begin{aligned}\{2f(f-c)-ml\}(X^2+Y^2+Z^2)+2Z\{(f-c)(-2f^2+K^2)+fml\}&\\ +\{2f(f-c)(f^2-K^2)-f^2ml+K^4\}&=0,\end{aligned}$$

which gives the spheres of the Inversion: the two equations take a more simple form if we write therein $K^2 = 2f^2$; viz. they then become

$$\alpha X + \beta Y - fZ = 0,$$
$$(2f^2 - 2fc - ml)(X^2 + Y^2 + Z^2) + 2Zfml + f^2(2f^2 + 2cf - ml) = 0;$$

or say these are

$$\left\{\begin{aligned} &\frac{\alpha X + \beta Y - fZ}{\sqrt{\alpha^2 + \beta^2 + f^2}} = 0, \\ &X^2 + Y^2 + Z^2 + \frac{2fml}{2f^2 - 2fc - ml} Z + \frac{f^2(2f^2 + 2cf - ml)}{2f^2 - 2fc - ml} = 0. \end{aligned}\right.$$

Interchanging the parameters so as to have t in the first equation and θ in the second equation, these are of the form

$$\begin{aligned} &aX + bY + cZ &&= 0 \quad (\text{where } a^2 + b^2 + c^2 = 1), \\ &X^2 + Y^2 + Z^2 - 2\gamma Z - 2v = 0, \end{aligned}$$

and the Inversion is thus a surface $PS3^0$.

$SS4^0$ = SERRET'S SECOND CASE OF SS.

Writing for convenience $m' = -mf$, $m''' = \frac{1}{2}(e^2 + f^2)$, $mm'' = -g$, and therefore $m'm'' = fg$, the six equations are

$$\begin{aligned} &A^2 + B^2 + C^2 &&= 1, \\ &x^2 + y^2 + z^2 - 2by - 2c(z - g) - 2fg + e^2 + f^2 = 0, \\ &Ax + B(y - b) + C(z - c) + m(f - c) &&= 0, \\ &x^2 + y^2 + z^2 - 2\alpha x - 2\gamma(z - f) - e^2 - f^2 &&= 0, \\ &A(x - \alpha) + By + C(z - \gamma) - \frac{1}{m}(g + \gamma) &&= 0, \\ &Adx + Bdy + Cdz &&= 0, \end{aligned}$$

where e, f, g, m are constants; b, c are functions of t; α, γ functions of θ.

The spheres of the first set pass all of them through the two points

$$x = \pm\sqrt{2fg - e^2 - f^2 - g^2}, \quad y = 0, \qquad z = g,$$

and those of the second set pass all of them through the two points

$$x' = 0, \qquad\qquad y' = \pm e, \quad z' = f,$$

where observe that these are such that

$$(x - x')^2 + (y - y')^2 + (z - z')^2 = 0;$$

viz. the distance of each point of the first pair from each point of the second pair is $= 0$. The pairs of points are one real, the other imaginary, but this is quite consistent with the reality of the spheres.

The first pair of points lie in a line parallel to the axis of x, meeting the axis of z at the point $z = g$; and the second pair in a line parallel to the axis of y, cutting the axis of z at the point $z = f$. It is clear that we can, without loss of generality, by moving the origin along the axis of z, in effect make g to be $= -f$; the equations of the two sets of spheres thus become

$$\begin{aligned} x^2 + y^2 + z^2 - 2by - 2c(z+f) + e^2 + 3f^2 &= 0, \\ x^2 + y^2 + z^2 - 2ax - 2\gamma(z-f) - e^2 - f^2 &= 0, \end{aligned}$$

or, if in these equations for e^2 we write $e^2 - 2f^2$, the equations become

$$\begin{aligned} x^2 + y^2 + z^2 - 2by - 2c(z+f) + e^2 + f^2 &= 0, \\ x^2 + y^2 + z^2 - 2ax - 2\gamma(z-f) - e^2 + f^2 &= 0, \end{aligned}$$

which are very symmetrical forms.

The spheres of the first set pass through the two points

$$\pm\sqrt{-e^2 - 2f^2}, \quad 0, \quad -f,$$

and those of the second set through the two points

$$0, \quad \pm\sqrt{e^2 - 2f^2}, \quad f,$$

where, of course, the two pairs of points are related as is mentioned above.

By taking as centre of inversion a point of the first pair, we invert the first set of spheres into planes and the second set into spheres; and similarly, by taking a point of the second pair, we invert the first set of spheres into spheres and the second set into planes. By reason of the symmetry of the system it is quite indifferent which point is chosen; and taking it to be a point of the second pair, and writing for convenience $n = \sqrt{e^2 - 2f^2}$ (n is in fact the quantity originally denoted by e), then the points of the first pair are

$$\begin{array}{ccc} \pm\sqrt{-n^2 - 4f^2}, & 0, & -f, \\ 0\quad, & \pm n, & f, \end{array}$$

and I take for centre of inversion the point $(0, n, f)$.

Observe that if $e = 0$, $f = 0$, then the four points coincide at the origin, and taking this as centre of inversion, the two sets of spheres are each changed into planes, and the Inversion of the surface SS is thus a surface PP; this particular case will be considered further on, but I first consider the general case.

The formulae of inversion are

$$x=\frac{K^2X}{\Omega},\quad y-n=\frac{K^2(Y-n)}{\Omega},\quad z-f=\frac{K^2(Z-f)}{\Omega},$$

where $\Omega=X^2+(Y-n)^2+(Z-f)^2$.

Writing the equation of the second set of spheres in the form

$$x^2+(y-n)^2+(z-f)^2-2ax+2n(y-n)+2(f-\gamma)(z-f)=0,$$

the transformed equation is

$$-aX+n(Y-n)+(f-\gamma)(Z-f)+\tfrac{1}{2}K^2=0,$$

which gives the planes of the Inversion.

Similarly writing the equation of the first set of spheres in the form

$$x^2+(y-n)^2+(z-f)^2+2(n-b)(y-n)+2(f-c)(z-f)+2n^2-2bn+4f(f-c)=0,$$

the transformed equation is

$$\begin{aligned}&\{n^2-bn+2f(f-c)\}\{X^2+(Y-n)^2+(Z-f)^2\}\\&\qquad+K^2\{(n-b)(Y-n)+(f-c)(Z-f)\}+\tfrac{1}{2}K^4=0,\end{aligned}$$

which gives the spheres of the Inversion.

Changing the origin, the two equations may be written

$$-aX+nY+(f-\gamma)Z+\tfrac{1}{2}K^2=0,$$
$$\{n^2-bn+2f(f-c)\}(X^2+Y^2+Z^2)+K^2\{(n-b)Y+(f-c)Z\}+\tfrac{1}{2}K^4=0.$$

I stop to consider a particular case: Suppose $n=0$, the equations are

$$-aX+(f-\gamma)Z+\tfrac{1}{2}K^2=0,$$
$$X^2+Y^2+Z^2-\frac{bK^2}{2f(f-c)}Y+\frac{K^2}{2f}Z+\frac{K^4}{4f(f-c)}=0,$$

or, interchanging herein Y and Z, they are

$$-aX+(f-\gamma)Y+\tfrac{1}{2}K^2=0,$$
$$X^2+Y^2+Z^2+\frac{K^2}{2f}Y-\frac{bK^2}{2f(f-c)}Z+\frac{K^4}{4f(f-c)}=0,$$

and if for Y we write $Y-\dfrac{K^2}{4f}$, then the equations become

$$-aX+(f-\gamma)Y+K^2\frac{f+\gamma}{4f}=0,$$
$$X^2+Y^2+Z^2-\frac{bK^2}{2f(f-c)}Z+\frac{K^4(5f-4c)}{f^2(f-c)}=0,$$

viz. interchanging the parameters so as to have t in the first equation and θ in the second equation, these are of the form

$$aX + bY = lm,$$
$$X^2 + Y^2 + Z^2 - 2\theta Z = 2v,$$

which belong to $PS4^0$. Hence in this particular case, $n = 0$, the Inversion is $PS4^0$.

Reverting to the general case, and to the two equations obtained above, observe that in the second of the two equations the terms in Y, Z have the variable coefficients $n - b$ and $f - c$, so that it does not at first sight seem as if these terms could be by a transformation of coordinates reduced to a single term.

But if again changing the origin we write $Y - \frac{\frac{1}{2}K^2}{n}$ for Y, the two equations become

$$- aX + nY + (f - \gamma) Z = 0,$$
$$\{n^2 - bn + 2f(f - c)\}\{X^2 + Y^2 + Z^2) + \frac{K^2}{n}(f - c)(-2fY + nZ)$$
$$+ \frac{K^4}{4n^2}\{n^2 + bn + 2f(f - c)\} = 0,$$

where, in the second equation, the terms in Y, Z present themselves in the combination $-2fY + nZ$ with the constant coefficients $-2f$ and n. Hence writing

$$\sqrt{n^2 + 4f^2}\, Y = nY' - 2fZ',$$
$$\sqrt{n^2 + 4f^2}\, Z = 2fY' + nZ',$$

and consequently $-2fY + nZ = \sqrt{n^2 + 4f^2}\, Z'$, and (after the transformation) removing the accents, the equations become

$$- aX + \frac{1}{\sqrt{n^2 + 4f^2}}[\{n^2 + 2f(f - \gamma)\} Y - n(f + \gamma) Z] = 0,$$
$$X^2 + Y^2 + Z^2 + \frac{K^2(f - c)\sqrt{n^2 + 4f^2}}{n\{n^2 - bn + 2f(f - c)\}} Z + \frac{K^4\{n^2 + bn + 2f(f - c)\}}{n^2\{n^2 - bn + 2f(f - c)\}} = 0,$$

viz. interchanging the parameters so as to have t in the first equation and θ in the second equation, these are of the form

$$aX + bY + cZ = 0,$$
$$X^2 + Y^2 + Z^2 - 2m\phi Z = \theta,$$

which belong to the case $PS3^0$. Hence in this general case the Inversion is a surface $PS3^0$.

I have spoken above of the particular case $e=0, f=0$: here the equations of the two sets of spheres are

$$\begin{aligned} x^2+y^2+z^2-2by-2cz&=0,\\ x^2+y^2+z^2-2ax-2\gamma z&=0, \end{aligned}$$

which have the origin as a common point. Taking this as the centre of inversion, or writing

$$x=\frac{K^2X}{\Omega},\quad y=\frac{K^2Y}{\Omega},\quad z=\frac{K^2Z}{\Omega},\ \text{where } \Omega=X^2+Y^2+Z^2,$$

the transformed equations are

$$\begin{aligned} bY+cZ-\tfrac{1}{2}K^2&=0,\\ aX\quad+\gamma Z-\tfrac{1}{2}K^2&=0, \end{aligned}$$

or, interchanging X and Y, say

$$\begin{aligned} bX\quad+cZ-\tfrac{1}{2}K^2&=0,\\ aY+\gamma Z-\tfrac{1}{2}K^2&=0, \end{aligned}$$

which are of the form

$$\begin{aligned} X\quad+tZ-P\quad&=0,\\ Y+\theta Z-\Pi\quad&=0, \end{aligned}$$

belonging to a surface $PP3^0$. Hence in this case the Inversion is a surface $PP3^0$.

It thus appears that the surface $SS4^0$ has an Inversion which is either $PS3^0$, $PS4^0$ or $PP3^0$. The inversion has in some cases to be performed in regard to an imaginary centre of inversion.

It was previously shown that the surface $SS3^0$ had an Inversion $PS3^0$, and we thus arrive at the conclusion that a surface SS, with its two sets of curves of curvature each spherical, is in every case the Inversion of a surface PS with one set plane and the other spherical, or else of a surface PP with each set plane. Serret notices that the centre of inversion may be imaginary: this (he says) presents no difficulty, but he adds that it is easy to see that the centres of inversion may be taken to be real, provided that we join to the surfaces thus obtained all the parallel surfaces.

It seems to me that there is room for further investigation as to the surfaces SS: first, without employing the theory of inversion, it would be desirable to obtain the several forms by direct integration, as was done in regard to the surfaces PP and PS; secondly, starting from the several surfaces PP and PS considered as known forms, it would be desirable to obtain from these, by inversion in regard to an arbitrary centre, or with regard to a centre in any special position, the several forms of the surfaces SS. But I do not at present propose to consider either of these questions.

In conclusion, I remark that I have throughout assumed Serret's *negative* conclusions, viz. that the several cases, other than those considered in the present memoir, give only developable surfaces, or else surfaces having for one set of their curves of curvature circles. These being excluded from consideration, there remain

PP, Serret's two cases $PP1^0$, $PP3^0$;
PS, his three cases $PS1^0$, $PS3^0$, $PS4^0$;
SS, his two cases $SS2^0$ and $SS4^0$;

but $PP1^0$ is a particular case of, and so may be included in, $PP3^0$; and similarly $PS1^0$ is a particular case of, and may be included in, $PS3^0$; the cases considered thus are

$$PP3^0;\ PS3^0,\ PS4^0;\ SS2^0 \text{ and } SS4^0.$$

It would however appear by what precedes that the case $SS4^0$ includes several cases which it is possible might properly be regarded as distinct; and the classification of the surfaces SS can hardly be considered satisfactory; it would seem that there should be at any rate 3 cases, viz. the surfaces which are the Inversions of $PP3^0$, $PS3^0$ and $PS4^0$ respectively.

I regard the present memoir as a development of the analytical theory of the surfaces $PP3^0$, $PS3^0$ and $PS4^0$.

On the Geometry of a Nodal Circular Cubic.

By F. Morley.

In Crelle Bd. V the circular cubic with double focus on itself is treated by Schröter and Durège. I here give a geometrical account of the case when the curve, in addition, is nodal. It is then (Humbert, *American Journal* X, 3, p. 279) Quetelet's "focale à nœud," but I have not been able to find Quetelet's work. Some properties of the special case when the inflexion is at infinity are given by Booth (Quarterly Journal, Vol. III) under the name of the logocyclic curve.

§1. Let two tangents OP, OQ be drawn to a conic U, and let a conic U' through OPQ meet U again at P_1Q_1. Then we know that the tangents at P_1Q_1 meet at a point O_1 on U'. In proof, project P_1Q_1 to the circular points on the line ∞; then we have tangents OP, OQ to a circle U, and the circle OPQ or U' obviously goes through the centre O_1 of this circle. Also OO_1 is conjugate to PQ and P_1Q_1 with regard to U'. We shall need the following Lemmas from the geometry of the circle:

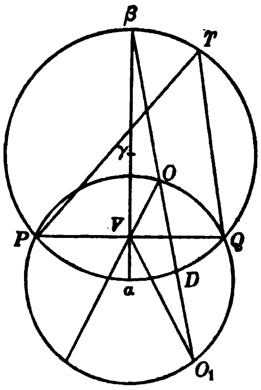

Fig. 1.

(1). Let OO_1, PQ be conjugate chords of a circle α, then the lines joining the ends of one to the centre of the other make with it equal angles. For let β be the pole of PQ. The centres of all chords through β lie on the circle γ whose diameter is $\alpha\beta$; hence if D be the centre of OO_1, the angle $PD\beta =$ angle $QD\beta$; also each is half angle $P\alpha Q$ and hence $= PO_1Q$.

(2). Since OV, O_1V are equally inclined to PQ at its centre (Lemma 1), they meet the circle in points equidistant from P, Q; hence angle $POV =$ angle QOD.

We may notice the corollaries that (1) the lines bisecting the angles POQ,

PO_1Q also bisect VOO_1, VO_1O, and hence by Lemma 1 meet on PQ, so that $PO:OQ = PO_1:O_1Q$; whence in a cyclic quadrilateral whose diagonals are conjugate, the rectangles of opposite sides are equal; (2) $OV.O_1V = PV^2$.

Returning to the conic U, let U' be a circle. We have the common chords PQ, P_1Q_1 of the circle and conic equally inclined to either axis, and therefore the radii CO, CO_1 equally inclined to either axis. Also from Lemma 2, angle $COP =$ angle O_1OQ, and therefore $COf_1 = O_1Of$. Hence the point O_1 has reference only to O, f, f_1, and is the same for all confocal conics when O is fixed; so that if we draw tangents from a fixed point O to a confocal family, the circles through O and the points of contact have a common radical axis (a result given in Wolstenholme, Problem 1079). The point O_1 clearly lies on the special circle Off_1, and also on the special circle (orthogonal to OFF_1) through O and the imaginary foci GG_1. We have of course $CO.CO_1 = Cf^2$.

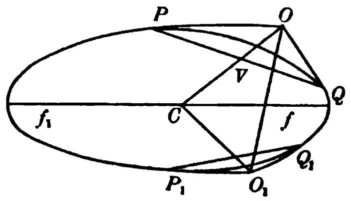

FIG. 2.

We know that the locus of P or Q is a circular cubic having a node at O, the nodal tangents bisecting the angle fOf_1; and that the cubic is also the locus of the feet of normals from O to the conics, and of feet of perpendiculars from O on its polars with regard to the conics. See a paper in the Messenger of Mathematics, April 1887, where I have given some of the geometry of this cubic, showing in particular that the tangents to it at P, Q meet at a point T on the cubic, so that PQ are "corresponding points"; that OP, OQ subtend equal or supplementary angles at any point of the curve (the fundamental property); in particular, that they make equal angles with either nodal tangent; that if PQ meets the cubic again at R, OR is perpendicular to PQ, and T, R are corresponding points, and that O is the centre of a circle touching TP, TQ, PQ (the latter of course at R). The angles which OP, OQ subtend are equal when R is between PQ, otherwise supplementary.

From what precedes, all circles through the node O and two corresponding points PQ pass through a fixed point O_1; and D being the centre of PO_1, OP, OQ subtend equal angles at D. Hence, from the fundamental property of the cubic, D is on the curve (Fig. 3). The line OV which bisects the lines joining corresponding points is parallel to the asymptote, for it is normal to the circle of infinitely large radius which is a limiting form of the confocal ellipses, i. e. it goes through the point at ∞ on the curve, K suppose. Now OV, OD make equal angles with a nodal tangent (Lemma 2); hence D, K correspond. But IJ being the circular points, the line at ∞ KIJ is perpendicular to any line and hence to VOK. Hence IJ correspond, and the tangents at them meet on the curve at D; or D is the double focus.

Any circle whose centre is D will have double contact with the cubic at IJ, for DI, DJ are tangents to both curves.

Since D, K correspond, angle $DQO =$ angle KQO, and if QD meets the cubic again at Q_1, angle $DQ_1O =$ angle KQ_1O. Hence angle QOQ_1 is right, or any chord through the double focus subtends a right angle at the node.

Let the line joining any point H on the curve to PQ meet the curve again at pq (see Fig. 6, where, however, H is special). Then, since angles OHP, OHQ are equal or supplementary, angles OHp, OHq are equal or supplementary, and therefore p, q also correspond. Conversely, the lines joining P, Q to p, q meet in two points on the curve which also correspond (see Salmon, Plane Curves, p. 132). The point O being equidistant from the lines joining corresponding points to any point on the curve (fundamental property) must be equidistant from the four lines Pp, Pq, Qp, Qq; or the four lines joining one pair of points to another pair touch a circle whose centre is the node, and intersect again on the curve (compare the paper above referred to).

In particular, QDQ_1 and PK (Fig. 3) meet on the curve, so that PQ_1 is parallel to the asymptote. Hence OV, which bisects PQ, also bisects QQ_1, or the locus of centres of chords through the double focus is the line through the node parallel to the asymptote.

The angle $PDQ =$ twice angle PO_1Q (Lemma 1) $= 360° -$ twice angle POQ. But O being centre of circle inscribed in TPQ, $POQ = 90° + \frac{1}{2} PTQ$. Hence $PDQ = 180° - PTQ$, or the circle circumscribed to the triangle formed by two tangents from a point on the curve, and the chord of contact, goes through the double focus. It is clear (see Fig. 1) that the tangents to the circle OPQ at P, Q meet on the circle TPQ. The locus of the centre of this latter circle may be seen to be a hyperbola of no apparent interest.

Let HH_1 be the chord through D perpendicular to OO_1. Since HH_1 is perpendicular to OD, HH_1 correspond. Hence OH, OH_1 make equal angles with the nodal tangents, and since OH, OH_1 are at right angles (HH_1 passing through D), they bisect the angles between the nodal tangents. Also, HH_1 being perpendicular to OD, the tangents at HH_1 are parallel to the asymptote. These points are of great use later.

Since D, K correspond, the tangent at D meets the asymptote on the curve, at A suppose. Since OD, OK subtend equal angles at D, the tangent and the asymptote are equally inclined to OD. Also AOD is a right angle since AD is a chord through the double focus. Hence if E is the centre of HH_1, $AODE$ is a rectangle.

§2. In connection with what follows see Salmon's Plane Curves, §§278–281. The bicircular quartic with a finite node being the pedal of a conic and the inverse of another conic, a circular cubic is the pedal of a parabola and the inverse of a conic with regard to a point on it, and the cubic in question having perpendicular nodal tangents, is the pedal of a parabola with regard to a point on the directrix and the inverse of a rectangular hyperbola with regard to a point on itself.

Since V is on a fixed line and PQ makes equal angles with OV, O_1V, we see that PQ envelopes a parabola with focus O_1 and directrix OV; and OR is

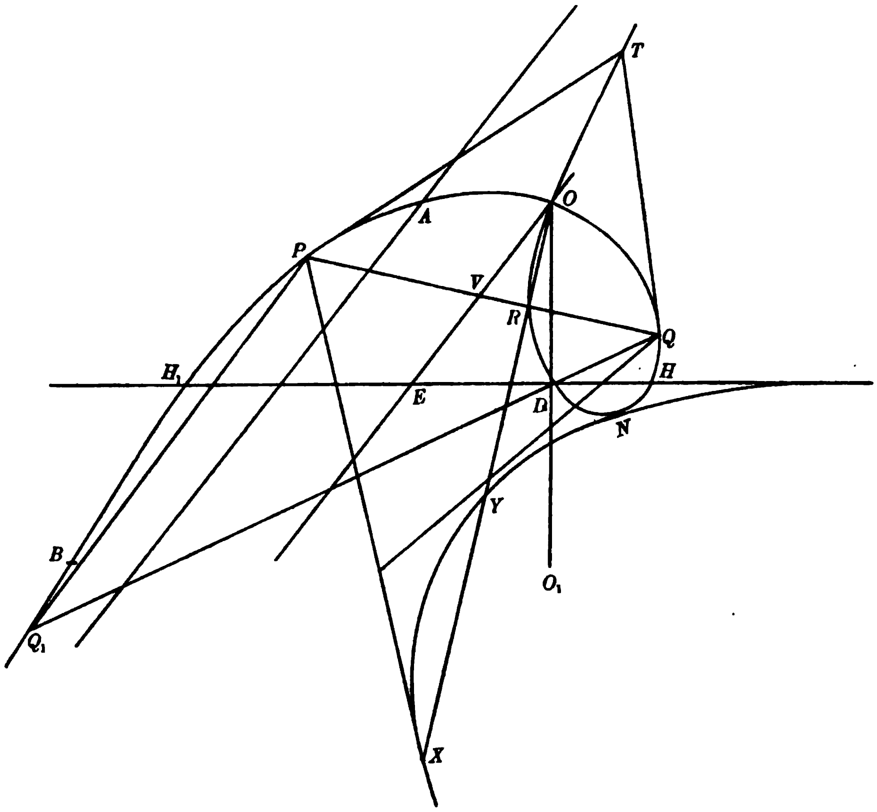

FIG. 8.

perpendicular to PQ. Thus we see *ab initio* that the curve is the pedal of a parabola. That it is the inverse of a hyperbola follows by reciprocation, or may be obtained directly from the fundamental property.

We now show that the tangents to the parabola from P, Q (which are perpendicular to OP, OQ) touch the parabola where OR meets it. It is enough to show that if PQ, PX be tangents to a parabola, the perpendiculars from X to PQ and from P to PX meet on the directrix. But these are perpendiculars of the triangle formed by PQ, PX and the tangent consecutive to PX. And we know that the orthocentre of any tangent triangle is on the directrix. It is obvious that PX, QY intersect on the polar of O, i. e. the line O_1B parallel to HH_1, and also on the circle $OPQO_1$. When QR coincide (at N in fig.), since the tangents at P, Q meet on the cubic and PQ is the tangent at Q, P must be the inflexion, and since it is the intersection of tangents from P, Q to the parabola, it must coincide with B, where the polar of O meets the cubic. Hence the inflexion and node are equidistant from HH_1. The curves clearly touch at N, and ON is the common normal. Since B, N correspond, OV bisects BN. Hence the tangent from the inflexion to the cubic is bisected by the parallel to the asymptote through the node.

If p, q correspond, then since OR, PQ make equal angles with Rp, Rq, they divide pq harmonically. Making pq coincide with PQ, the line PRQ is divided harmonically at the contact with the parabola.

It is worth while to call attention to the fact that D bisects OO_1. It may be proved (but is apart from the present purpose) that the four double foci of a nodal bicircular quartic bisect the lines joining the node to the foci of the conic which is the negative pedal of the quartic.

§3. We will now obtain the position of the four single foci of the cubic, by considering the cubic as the inverse of a rectangular hyperbola with regard to a point O on it.

The tangent at O inverts into a parallel to the finite asymptote. The parallels to the asymptotes through O become the nodal tangents. The four foci of the hyperbola become the four foci of the cubic. The following consideration may aid in clearing up this last point, which is usually stated without proof. Let O be the origin, and let a point F, whose coordinates are a, 0, be a focus of a curve. Let F_1G_1 be antipoints of O, F; the coordinates of F_1 are $a/2$, $\iota a/2$.

The circle OFF_1 is $x^2 + y^2 - ax - \iota ay$, or $(x + \iota y)(x - \iota y - a) = 0$, i. e. it breaks up into the imaginary lines OF_1I, F_1FJ. Since the latter is a tangent to the curve, so also is the circle. Since $OF_1^2 = 0 = OG_1^2$, when we invert the antipoints and the circular points are interchanged, the circle becomes a tangent through a circular point, and F remains a focus of the inverse curve.

Since the tangents at corresponding points of a curve and its inverse make equal angles with the radius vector, the images of O with regard to the axes will become the points HH_1 at which the tangents are parallel to the asymptote.

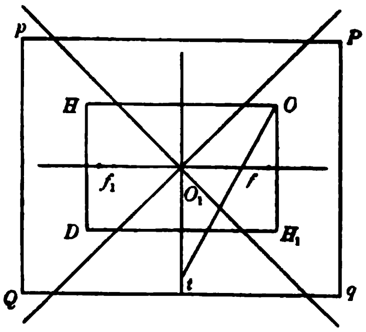

FIG. 4.

The end of the diameter through O lies on the circle OHH_1 and hence becomes the point where HH_1 cuts the cubic, that is the double focus. The ends of any diameter become corresponding points.

The foci ff_1 of the hyperbola are on the axis ff_1, which inverts into a circle through O with centre H_1. They are also on the circle Off_1H. If the tangent at O meets the conjugate axis at t, then the angle which the circle makes with $Ot =$ angle $tHO =$ angle tOH; hence in the cubic the foci lie on the line through H which makes with the asymptote the same angle as OH, and are the points where this line is cut by the circle through O with centre H_1.

Interchanging H and H_1, we have the imaginary foci, the antipoints of the real foci.

Since (Fig. 5) H is the pole of OO_1 with regard to the circle whose centre is H_1 and radius HO or HO_1, we see that OO_1 and ff_1 are conjugate chords. Hence DOf, f_1DO are similar triangles, and (cor. 2 to Lemma 2 at the beginning) $OD^2 = Df.Df_1$, a result generalized later.

It is clear that the focal circle with centre H_1 and radius H_1O, being the inverse of an axis, cuts the cubic orthogonally at points where the curvature is stationary.

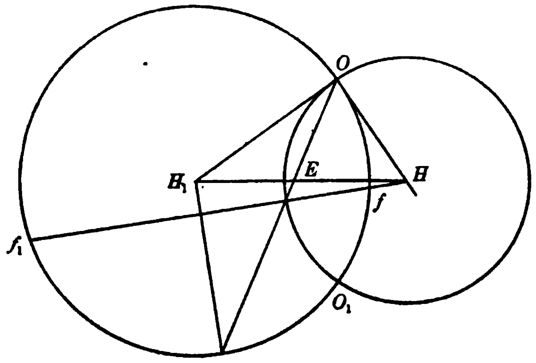

FIG. 5.

In the hyperbola we have $fP - f_1P = \pm k$, — when P is on the same branch as O. Hence on inversion

$$\frac{fP}{Of.OP} - \frac{f_1P}{Of_1.OP} = \pm k,$$

$+$ for the loop, because the branch of the hyperbola away from O inverts into the loop; f_1 is now the focus inside the loop. To determine k, take P at H_1 where $H_1f = H_1f_1 = H_1O$; then

$$\frac{1}{Of} - \frac{1}{Of_1} = -k,$$

and hence
$$\frac{fP \pm OP}{Of} = \frac{f_1P \pm OP}{Of_1},$$

where for points on the loop we take the $+$ sign.

Again in the hyperbola we have $Pf.Pf_1 = PO_1^2$, $Of.Of_1 = OO_1^2$, and hence on inversion
$$\frac{Pf}{OP.Of}\cdot\frac{Pf_1}{OP.Of_1} = \left(\frac{PO_1}{OP.OO_1}\right)^2, \; Of.Of_1 = OO_1^2,$$

whence
$$Pf.Pf_1 = PO_1^2,$$

so that the cubic is the locus of a point whose distances from 3 fixed points are connected by the relation $r_1r_2 = r_3^2$. It is to be noticed that O_1 is not a focus. In the linear relation above connecting the distances from f, f_1, O, we may regard O as an improper focus, for the line from it to a circular point fulfils the condition for a tangent.

From the general case, when we take any conic and a point off it, we get that the product of focal distances of P is in a constant ratio to the rectangle of segments of a chord from P to a fixed circle, namely, the inverse of the director circle. The ratio becomes an equality when the fixed point is on the conic, in which case the quartic becomes a cubic.

From the equation $Pf.Pf_1 = PO_1^2$ we get that the line of foci meets the cubic at points on the circle H_1DO_1 as well as at H.

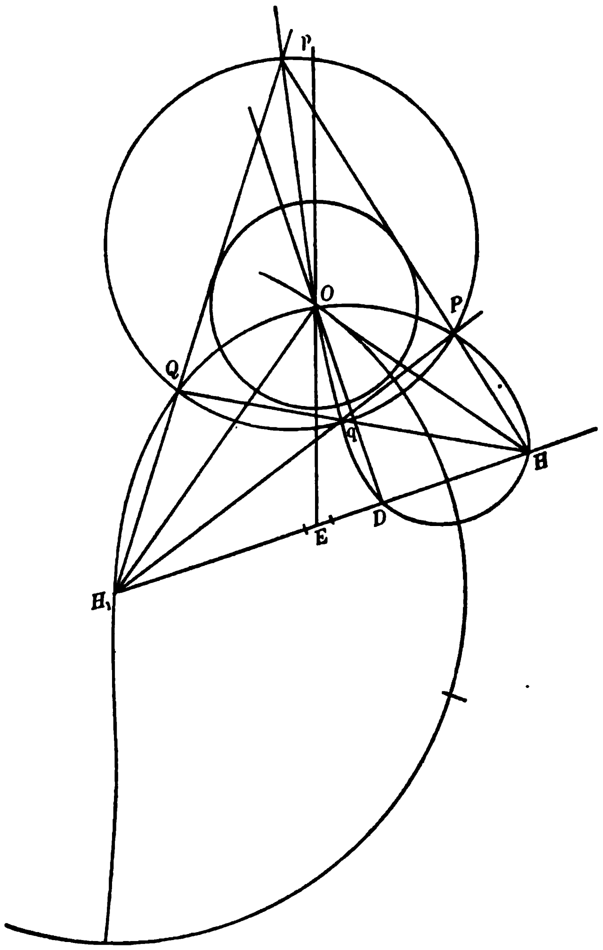

FIG. 6.

The cubic is its own inverse with regard to H or H_1 (Fig. 6). For let HPp be a chord through H. We have to show $HP.Hp = HO^2$. Inverting we have

(Fig. 4) a circle through OH, and Pp are clearly on a line parallel to ff_1. We have to show in the hyperbola

$$\frac{HP}{OH.OP}\cdot\frac{Hp}{OH.Op}=\frac{1}{HO^2},$$

or

$$HP.Hp=OP.Op,$$

which is obvious. This clearly applies to any nodal cubic.

Join Pp to H_1, cutting the cubic at qQ. We have shown above that PQ, pq (since they join corresponding points to a point on the curve) are corresponding points, that Qq goes through H, and that the four lines touch a circle with centre O. Also, since $HP.Hp=HQ.Hq=HO^2$, the four points lie on a circle orthogonal to the circles whose centres are H, H_1 and radii HO, H_1O; hence the centre of this circle is on OD, the radical axis of these circles. Since the circle OPQ has its centre on HH_1, it is orthogonal to the circle pPQ; the former belongs to a coaxial set through OO_1, the latter to the orthogonal coaxial set whose limiting points are OO_1. The line joining the centres of these circles, being perpendicular to PQ at its centre V, will touch the parabola of Fig. 3, for V is on the directrix. It therefore joins corresponding points, but they are imaginary, being in fact the antipoints of PQ. If it meets the cubic at r, clearly Or is a perpendicular on it, and since ROr is a right angle (Fig. 3), Rr goes through D and bisects OV. We thus have what is probably the simplest geometrical definition. Let O be a fixed point. Take Z on a fixed line through it, join Z to a fixed point D, and take R on ZD such that $ZR=ZO$. The locus of R is a right circular cubic of which O is node, OZ parallel to the asymptote, D double focus. This is the simplest starting-point for geometrical treatment.

Returning to Fig. 6, we may see by inversion that the diagonals PQ, pq meet on OD. This is, however, the known fact that when a quadrilateral is inscribed in one circle and described about another, the diagonals (and also the lines joining opposite points of contact) meet at a point on the line of centres. See Casey's "Sequel to Euclid," pp. 108, 94.

We see that the cubic is the locus of intersections of tangents from fixed points to circles with a given centre. The quadrilateral formed by the tangents is cyclic when the points subtend a right angle at the centre. For these circles with centre at the node, in connection with the theory of confocal conics, see the paper in the Messenger of Math. above referred to.

§4. The *raison d'être* of the above treatment of the *focale à nœud* is that any projective theorem for a nodal cubic or tricuspidal quartic becomes intuitive by projection or reciprocation. The leading feature of the geometry is the parabola of Fig. 3. Bearing in mind that the polar of the node with regard to this parabola will contain the two imaginary inflexions as well as the real one, we see that in a nodal cubic the envelope of the line joining corresponding points is a conic touching the cubic at the three sextactic points, and touching the nodal tangents at points on the line of inflexions, so that the node and line of inflexions are pole and polar with regard to it. It is the Cayleyan of the cubic of which the given cubic is the Hessian. Salmon, §177, and Durège, Curven dritter Ordnung, §540.

The following may be instanced as properties immediately following from the *focale à nœud:*

(1). Any two points on a nodal cubic and the points of contact of tangents from them lie on a conic. For the circle TPQ (Fig. 3) goes through D.

(2). Let PQ be corresponding points, T their tangential, R the point on the curve collinear with PQ, O the node. Any line through T, or any line joining corresponding points, is cut harmonically by OR, PQ, and the curve. If qTq_1, qp are such lines, pq_1 goes through R.

(3). If we draw tangents RX, RY, join XY to corresponding points xy, meeting the curve at corresponding points $\xi\eta$, then $PQxy\xi\eta$ lie on a conic, and a conic will pass through PQ and touch the lines joining XY to xy.

(4). The line OT is cut harmonically by the line of inflexions and the line PQ.

(5). A complete quadrilateral is inscribed in a cubic whose node is O. The 3 diagonals form a triangle such that the lines from O to the angles meet the sides on the cubic.

(6). Four lines meet at $AA_1BB_1CC_1$. Conics through AA_1BB_1 and AA_1CC_1 meet again at OO_1. With OO_1 as nodes, cubics are drawn through $AA_1BB_1CC_1$. Then they touch at AA_1, the tangents meet on the curve and on OO_1, at a point which with AA_1 cuts OO_1 harmonically; a conic through AA_1OO_1, with regard to which AA_1 and OO_1 are conjugate, will touch the cubics at AA_1.

(7). In the cardioid a tangent meets the curve at two points, the tangents at which meet the tangent at the vertex at points equidistant from the vertex.

Results in relation to some here considered are given by McIntosh (Educational Times Reprint, XXXVIII, p. 82) and Wolstenholme (Ib. XLII, p. 81, and XLIII, p. 77, and Problems, Nos. 1824, 1850, 1841).

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On the Functions Defined by Differential Equations, with an Extension of the Puiseux Polygon Construction to these Equations.

By Henry B. Fine.

In their memoir *Propriétés des fonctions définies par des équations différentielles* (Journal de l'Ecole Pol. Cah. 36) Briot and Bouquet present methods for obtaining developments for all ordinary solutions of a differential equation of the first order $f(x, y, p) = 0$ which belong to the initial values $x = y = 0$ of the variables, when it is known that p as well as y vanishes with x.

But as a general equation $f(x, y, p) = 0$ which has no term independent of x, y, or p may very well have groups of terms of lower degree in respect to x than the remaining terms of the equation, for which p does not vanish with x, but remains finite and different from zero, or becomes infinite—and yet the corresponding y does vanish and is a solution of the equation,—it often becomes necessary at the very outset, and directly from the differential equation itself, to make a determination of all the possible groups of terms of lowest dimension, and this Briot and Bouquet give no means of doing.

In the first section of the present paper it is shown how this determination may be very simply accomplished by an extension of the polygon construction used by Puiseux in his study of algebraic functions.* This done, the reduction of the cases where p does not vanish with x to the case so fully discussed by Briot and Bouquet is easy.

In the second section, after extending the polygon construction to the equation of the n^{th} order, and so determining for it also the terms of lowest dimension when the equation contains no term independent of x, y, or one of the differential coefficients $y_1, y_2, \ldots y_n$,—I give methods for obtaining the corresponding developments themselves. It is proven also that these developments are actual as well as formal solutions of the equation, for it is shown that they converge for values of x which are greater than zero.

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* Journal de Math. pure et appliquées, I, 15.

The method, of course, leads not only to all "monodrome" integrals which vanish with x, but to those also which belong to any initial values x^0, y^0, $y_1^0, \ldots y_{n-1}^0$ of the variables and the differential coefficients of lower orders, since the equation may in this case be readily transformed into one which has no absolute term.

§1.

Let $$f(x, y, p) \equiv \Sigma A_i x^{\alpha_i} y^{\beta_i} p^{\gamma_i} = 0$$

be any equation of the first order which has no term independent of x, y, or p.

It is proposed to make every determination of the terms of lowest degree in $\Sigma A_i x^{\alpha_i} y^{\beta_i} p^{\gamma_i} = 0$ which is possible on the assumption that y vanishes with x.

In every case represent by μ the degree of y in respect to x; by the hypothesis that y vanishes with x, $\mu > 0$.

Let $A_1 x^{\alpha_1} y^{\beta_1} p^{\gamma_1}$ be one of the required terms of lowest degree; there must be at least one other term, say $A_2 x^{\alpha_2} y^{\beta_2} p^{\gamma_2}$, of the same degree, and a comparison of the two gives for the corresponding μ the equation

$$\alpha_1 + \mu\beta_1 + (\mu - 1)\gamma_1 = \alpha_2 + \mu\beta_2 + (\mu - 1)\gamma_2,$$

whence $$\mu = -\frac{\alpha_1 - \gamma_1 - (\alpha_2 - \gamma_2)}{\beta_1 + \gamma_1 - (\beta_2 + \gamma_2)}.$$

Take two rectangular axes η, ξ, and construct a point $\xi_i = \alpha_i - \gamma_i$, $\eta_i = \beta_i + \gamma_i$ to correspond to each term $A_i x^{\alpha_i} y^{\beta_i} p^{\gamma_i}$. Then the line joining $\xi_1\eta_1$ and $\xi_2\eta_2$, viz. the line whose equation is

$$\xi - \xi_1 = \frac{\alpha_1 - \gamma_1 - (\alpha_2 - \gamma_2)}{\beta_1 + \gamma_1 - (\beta_2 + \gamma_2)}(\eta - \eta_1) = -\mu(\eta - \eta_1)$$

makes with the η-axis an angle of which the tangent is $-\mu$, and cuts off on the ξ-axis an intercept

$$\xi_1 + \mu\eta_1 = \alpha_1 - \gamma_1 + \mu(\beta_1 + \gamma_1),$$

which is equal to the common degree of the two terms $A_1 x^{\alpha_1} y^{\beta_1} p^{\gamma_1}$, $A_2 x^{\alpha_2} y^{\beta_2} p^{\gamma_2}$.

Since $\mu > 0$ the line makes an oblique angle with the η-axis.

Furthermore, a parallel to this line through any of the other points ξ_i, η_i cuts off on the ξ-axis an intercept $\alpha_i - \gamma_i + \mu(\beta_i + \gamma_i)$, equal to the degree of the corresponding term. If, therefore, $A_1 x^{\alpha_1} y^{\beta_1} p^{\gamma_1}$, $A_2 x^{\alpha_2} y^{\beta_2} p^{\gamma_2}$ be, as was supposed, terms of lowest degree, all the other points must lie to the same side of the line $\xi_1\eta_1 - \xi_2\eta_2$ as the origin when its intercept is negative, to the opposite side when its intercept is positive.

Hence to get every admissible group of lowest terms in the equation $\Sigma A_i x^{\alpha_i} y^{\beta_i} p^{\gamma_i} = 0$—that is, every group of terms for which the corresponding μ is positive and such as to make the terms of the group of lower degree than the remaining terms of the equation—move up a parallel to the η-axis from a position below any of the $\xi_i\eta_i$ points until it meets one of these points or a group of them; next turn it (clockwise, since $\mu > 0$) about the point of this group which is nearest the ξ-axis until it meets a second point or group of points; again turn it about the point of this second group which is nearest the ξ-axis, and so on until further turning would bring it past the position of parallelism with the ξ-axis.

To each side of the polygon thus constructed—except that parallel to the η-axis, should it occur—correspond one or more developments of y in increasing powers of x, each beginning with the term x^μ and—save in exceptional cases—satisfying the equation $f(x, y, p) = 0$.

A side parallel to the η-axis is to be rejected, since for it $\mu = 0$, or the corresponding y does not vanish with x.

The construction can make a parallel to the ξ-axis a polygon side only in case there be no mere x term in the equation—when $y = 0$ is a solution. Not all equations, however, of which $y = 0$ is a solution have this line for a polygon side; the equation $p + py + y^3 = 0$ does not, for instance.

The developments corresponding to any polygon side for which $\mu > 1$ can be obtained immediately by the Briot-Bouquet methods already referred to, since here p as well as y vanishes with x.

Those corresponding to a side for which $\mu < 1$ are to be obtained by the same methods after an interchange has been made of the dependent and independent variables.

If for any side $\mu = 1$, make in $f(x, y, p) = 0$ the substitutions

$$y = vx, \, p = v + \frac{dv}{dx} x,$$

where v takes a finite value v_0 different from zero when $x = 0$.

The values of v_0 belonging to the various developments are given by the group of terms of lowest order in $f(x, y, p) = 0$, to which the side $\mu = 1$ belongs, and by the substitution $y = (v_0 + y')x$, $f = 0$ is again reduced to the Briot-Bouquet form.

The various developments in all these cases may also be obtained by making in $f=0$ the substitutions $x=t^s$, $y=t^r v$, $p=\frac{t^{r-s}}{s}\left(rv+t\frac{dv}{dt}\right)$,—where $\frac{r}{s}=\mu$,—and determining the corresponding v's from the transformed equation. But this method is less direct and elegant than that of Briot-Bouquet.

§2.

It is obvious that the polygon construction is equally applicable to equations of higher orders.

The equation being $\Sigma x^{\alpha_\iota} y^{\beta_\iota} y_1^{\gamma_\iota} \dots y_n^{\nu_\iota} = 0$, supposed to have no term independent of x, y, ... or y_n, take as before axes of ξ and η, and following the method already explained for the equation of the first order, construct points

$$\xi_\iota = \alpha_\iota - \gamma_\iota - 2\delta_\iota \dots - n\nu_\iota, \quad \eta_\iota = \beta_\iota + \gamma_\iota + \delta + \dots \nu_\iota$$

to correspond to the various terms $A_\iota x^{\alpha_\iota} y^{\beta_\iota} \dots y_n^{\nu_\iota}$ of the equation. The polygon may then be constructed exactly as for the equation of the first order, and similar inferences drawn with reference to the terms of lowest order in the equation.

In the case of the equation of the n^{th} order it is necessary, when deriving the developments corresponding to the various polygon sides, to distinguish between sides whose μ is $> n$ and those whose μ is $=$ or $< n$. For the first y_n as well as y and all its lower differential coefficients vanish with x; for the second, y_n takes a finite value different from zero or becomes infinite when x vanishes.

It is always possible, however, by a simple transformation, to reduce the second case to the first. For if $\mu = n$, set

$$y = vx^n, \quad y_1 = nvx^{n-1} + x^n\frac{dv}{dx}, \text{ etc.}$$

in the group of lowest terms under consideration, remove the common factor x^n and set the group equal to zero. The roots v_0 of the equation thus constructed are the initial values of v in the set of developments sought for; and the substitution $y = x^n(v_0 + y')$—where y' vanishes with x—effects the required transformation.

If, on the other hand, $\mu < n$ for any side or series of sides, select the side whose μ is least, find m, the first integer greater than the quotient of n by this μ, and make the substitution $x = x'^m$. Then for all the groups of lowest terms in the transformed equation μ', the degree of y in respect to x', is greater than n.

It only remains therefore to indicate a general method for getting the various developments corresponding to a polygon side for which $\mu > n$.

This done, it may be added, we are in position to derive all solutions of the equation

$$f \equiv f_0 y_n^m + f_1 y_n^{m-1} + \dots + f_m = 0$$

which belong to *any* given initial values $x_0, y_0, \dots y_{n-1}^0$ of x, y and the lower differential coefficients of y. $f_0, f_1, \dots f_m$ are supposed to be holomorphic functions of $x, y, \dots y_{n-1}$ for a common region of convergence, and $x_0, y_0, \dots y_{n-1}^0$ to lie within this region.

For if f_m does not vanish for $x = x_0$, $y = y_0$, etc., the corresponding initial values y_n^0 of y_n may, unless f_0 vanishes, be immediately obtained from $f = 0$, which is algebraic in y_n; and the substitutions

$$x = x_0 + x',$$
$$\frac{d^\kappa y}{dx^\kappa} = y_\kappa^0 + y_{\kappa+1}^0 x' + y_{\kappa+2}^0 \frac{x'^2}{2!} + \dots y_n^0 \frac{x'^{n-\kappa}}{(n-\kappa)!} + y'_\kappa, \qquad (\kappa = 0,\ 1, \dots n)$$

transform $f = 0$ into an equation in $x', y', y'_1, \dots y'_n$ which has no term independent of one or other of these variables, and for which also y'_n, as well as the lower differential coefficients of y', vanishes with x'.

If, on the other hand, f_0 vanishes, apply the polygon construction already described to $f = 0$, regarded as an equation in $x_0 + x'$, $y_0 + y'$, $y_1^0 + y'_1, \dots y_{n-1}^0 + y'_{n-1}$, and $\frac{1}{y_n}$ (it has no term which does not involve one of these variables), and the various values of μ having been obtained, transform as above, by the substitution $x = x'^m$, into an equation whose n^{th} differential coefficient vanishes with x'.

To get the developments corresponding to a polygon side for which $\mu > n$, that is in the case where all the differential coefficients $y_1, y_2, \dots y_n$ as well as y vanish with x, make in the equation the substitutions

$$\left.\begin{aligned} y_{n-1} &= y_n v_1 x, \\ y_{n-2} &= y_n v_1 v_2 x^2, \\ &\dots\dots\dots\dots \\ y &= y_n v_1 v_2 \dots v_n x, \end{aligned}\right\} \tag{1}$$

where $v_1, v_2, \dots v_n$ are functions of x which take finite values $v_1^0, v_2^0, \dots v_n^0$ different from zero when $x = 0$.

In the resulting equation, freed from any factor $y_n^\alpha x^\beta v_1^\gamma \dots v_{n-1}^\nu$ which may be common to all the terms, determine by the ordinary Puiseux method or any other method applicable to algebraic functions of a single variable, the various groups of terms of lowest order in y_n and x: if indeed this determination has not already been made directly from the equation.

Suppose that for any particular group of terms of lowest order the degree of y_n in respect to x is $\frac{r}{s}$: to obtain the corresponding developments make then the further substitutions

$$x = x'^s,\ y_n = Vx'^r, \tag{2}$$

where again V takes an initial value V_0 different from 0 and ∞ when x' vanishes.

Since $y_n = V_0 x^{\frac{r}{s}}$ approximately, $y_{n-1} = \frac{s}{r+s} V_0 x^{\frac{r+s}{s}} + \dots\,$; but

$$y_{n-1} = y_n v_1 x = V_0 v_1^0 x^{\frac{r+s}{s}} + \dots\,;$$

therefore, by a comparison of the two values of y_{n-1}, $v_1^0 = \frac{s}{r+s}$. In like manner $v_2^0 = \frac{s}{r+2s}, \dots\, v_n^0 = \frac{s}{r+ns}$.

In the V equation therefore, make the substitutions

$$v_1 = \frac{s}{r+s} + v_1', \quad v_2 = \frac{s}{r+2s} + v_2', \dots\, v_n = \frac{s}{r+ns} + v_n' \tag{3}$$

when V will be given as a function of x', $v_1', \dots\, v_n'$, developable in integral powers of these variables.

The various developments for V having been obtained, the equations for the corresponding sets of values of $v_1', v_2', \dots v_n'$ are readily constructed by aid of the equations (1), (2), (3). For evidently on introducing the substitutions (2) in equations (1) we have:

$$Vx'^r = \frac{d}{dx}\left(Vv_1 x'^{r+s}\right),$$

$$Vv_1 x'^{r+s} = \frac{d}{dx}\left(Vv_1 v_2 x'^{r+2s}\right),$$

$$\dots\dots\dots\dots\dots\dots$$

$$Vv_1 v_2 \dots v_{n-1} x'^{r+(n-1)s} = \frac{d}{dx}\left(Vv_1 v_2 \dots v_n x'^{r+ns}\right),$$

or actually effecting the differentiations indicated, the equations

$$\left.\begin{aligned} & x'\frac{dv_1'}{dx'}\left[v_1\frac{\partial V}{\partial v_1'}+V\right]+x'\frac{dv_2'}{dx'}v_1\frac{\partial V}{\partial v_2'}+\ldots+x'\frac{dv_n'}{dx'}v_1\frac{\partial V}{\partial v_n'} \\ & \qquad =-Vv_1'(r+s)-v_1x'\frac{\partial V}{\partial x'}. \\ & x'\frac{dv_2'}{dx'}=\frac{s(v_1-v_2-v_1v_2)}{v_1}, \\ & x'\frac{dv_3'}{dx'}=\frac{s(v_2-v_3-v_2v_3)}{v_2}, \\ & \ldots\ldots\ldots\ldots\ldots\ldots \\ & x'\frac{dv_n'}{dx'}=\frac{s(v_{n-1}-v_n-v_{n-1}v_n)}{v_{n-1}}. \end{aligned}\right\} \quad (4)$$

On substituting for $x'\frac{dv_2'}{dx'}, \ldots, x'\frac{dv_n'}{dx'}$ in the first of these equations the functions of the v_i's to which the remaining equations declare them equal, and making the substitutions (3) in all the equations, the set is reduced to the form

$$\left.\begin{aligned} & x'\frac{dv_1'}{dx'}=f_1(x', v_1', v_2', \ldots v_n'), \\ & x'\frac{dv_2'}{dx'}=f_2(v_1', v_2'), \\ & \ldots\ldots\ldots\ldots\ldots\ldots \\ & x'\frac{dv_n'}{dx'}=f_n(v_{n-1}', v_n'), \end{aligned}\right\} \quad (5)$$

where $f_1, f_2, \ldots. f_n$ vanish with $x', v_1', \ldots. v_n'$, and are holomorphic functions of these variables for finite regions of convergence about $x'=0$, $v_1'=0$, etc.

Developments for $v_1', v_2', \ldots.$ in integral powers of x', which formally satisfy these equations, are to be obtained by differentiating them a first time, a second time, etc., successively, and after each differentiation making $x'=0$ and solving the resulting equations for the corresponding differential coefficients of the v_i''s in respect to x'.

Thus differentiating a single time, we have

$$\begin{aligned} \left(\frac{dv_1'}{dx'}\right)_0 &= \left(\frac{\partial f_1}{\partial x'}+\frac{\partial f_1}{\partial v_1'}\frac{dv_1'}{dx'}+\frac{\partial f_1}{\partial v_2'}\frac{dv_2'}{dx'}+\ldots.\frac{\partial f_1}{\partial v_n'}\frac{dv_n'}{dx'}\right)_0, \\ \left(\frac{dv_2'}{dx'}\right)_0 &= \left(\frac{\partial f_2}{\partial v_1'}\frac{dv_1'}{dx'}+\frac{\partial f_2}{\partial v_2'}\frac{dv_2'}{dx'}\right)_0, \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \left(\frac{dv_n'}{dx'}\right)_0 &= \left(\frac{\partial f_n}{\partial v_{n-1}'}\frac{dv_{n-1}'}{dx'}+\frac{\partial f_n}{\partial v_n'}\frac{dv_n'}{dx'}\right)_0; \end{aligned}$$

a set of linear equations in $\left(\frac{dv_1'}{dx'}\right)_0$, $\left(\frac{dv_2'}{dx'}\right)_0$, etc., which give finite determinate values for these quantities except when the determinant of their coefficients vanishes.

A second differentiation gives the equations

$$2\left(\frac{d^2v_1'}{dx'^2}\right)_0 = \left(\phi_1 + \frac{\partial f_1}{\partial v_1'}\frac{d^2v_1'}{dx'^2} + \frac{\partial f_1}{\partial v_2'}\frac{d^2v_2'}{dx'^2} + \ldots . \frac{\partial f_1}{\partial v_n'}\frac{d^2v_n'}{dx'^2}\right)_0,$$

$$2\left(\frac{d^2v_i'}{dx'^2}\right)_0 = \left(\phi_i + \frac{\partial f_i}{\partial v_{i-1}'}\frac{d^2v_{i-1}'}{dx_2'} + \frac{\partial f_i}{\partial v_2'}\frac{d^2v_i'}{dx'^2}\right)_0, \qquad i = 2, 3, \ldots . n$$

where ϕ_1^0, ϕ_i^0 are functions of $\left(\frac{dv_1'}{dx'}\right)_0$, $\left(\frac{dv_2'}{dx'}\right)_0$ etc., and of second differential coefficients of the f's; and from these equations the values of $\left(\frac{d^2v_1'}{dx}\right)_0$, $\left(\frac{d^2v_2'}{dx}\right)_0$, etc., may be reckoned.

Further differentiations give in like manner the higher differential coefficients. It is only necessary, therefore, for the construction of series

$$v_i' = \left(\frac{dv_i'}{dx'}\right)_0 x' + \frac{1}{2!}\left(\frac{d^2v_i'}{dx'^2}\right)_0 x'^2 + \frac{1}{3!}\left(\frac{d^3v_i'}{dx'^3}\right)_0 x'^3 + \ldots . , \qquad i = 1, 2, \ldots . n \qquad (6)$$

which *formally* satisfy the equations, that the determinant of the coefficients in each of these sets of equations shall be different from 0, or that

$$\left|\begin{array}{lllll}
\frac{\partial f_1}{\partial v_1'} - x, & \frac{\partial f_1}{\partial v_2'}, & \frac{\partial f_1}{\partial v_3'} & , \ldots\ldots\ldots\ldots, & \frac{\partial f_1}{\partial v_n'} \\
\frac{\partial f_2}{\partial v_1'}, & \frac{\partial f_2}{\partial v_2'} - x, & 0 & , \ldots\ldots\ldots\ldots, & 0 \\
0 & \frac{\partial f_3}{\partial v_2'}, & \frac{\partial f_3}{\partial v_3'} - x, & 0 \ldots\ldots\ldots, & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & , \ldots\ldots \frac{\partial f_n}{\partial v_{n-1}}, & \frac{\partial f_n}{\partial v_n} - x
\end{array}\right|_{x=0},$$

or

$$\left.\begin{array}{l}
V_0(r+s+x)(r+2s+x)\ldots.(r+ns+x) \\
+\frac{\partial V_0}{\partial v_1^0}\frac{xs}{r+s}(r+2s+x) \quad \ldots.(r+ns+x) \\
+\frac{\partial V_0}{\partial v_2^0}\frac{xs(r+s)}{r+2s}(r+3s+x)\ldots.(r+ns+x) \\
+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
+\frac{\partial V_0}{\partial v_n^0}\frac{xs(r+s)(r+2s)\ldots.r+(n-1)s}{r+ns}
\end{array}\right\} \qquad (7)$$

shall vanish for no positive integral value of x.

It may be added that when the determinant vanishes, the equations have no "monodrome" integrals unless $\left(\frac{\partial f_1}{\partial x}\right)_0 = 0$; but if $\left(\frac{\partial f_1}{\partial x}\right)_0 = 0$, an infinite number of such integrals. Also, that when the determinant does not vanish, there will be in certain cases, besides the "monodrome" integrals, an infinite number of "non-monodrome" integrals. The consideration of these integrals, however, is aside from the purpose of the present paper.*

It only remains to prove that the series (6) have circles of convergence whose radii are greater than zero.

To give the demonstration as general a character as possible, consider the system of equations:

$$\left.\begin{aligned} x\frac{dv_1}{dx} &= f_1(x, v_1, v_2, \ldots . v_n), \\ x\frac{dv_2}{dx} &= f_2(x, v_1, v_2, \ldots . v_n), \\ & \ldots\ldots\ldots\ldots\ldots\ldots \\ x\frac{dv_n}{dx} &= f_n(x, v_1, v_2, \ldots . v_n), \end{aligned}\right\} \tag{8}$$

where $f_1, f_2, \ldots . f_n$ are any set of functions of $x, v_1, v_2, \ldots . v_n$, developable in series of integral powers of these variables which vanish for $x = v_1 = v_2 = \ldots . v_n = 0$, but converge so long as mod $x < \rho$ and mod $v_i < r_i$, the quantities ρ and r_i being all greater than zero.

Differentiating each equation κ times successively with respect to x, and after each differentiation placing $x = 0$ and reckoning out the values of the differential coefficients of corresponding order of the v_i's, we have finally for the determination of the coefficients of the κ^{th} order the equations

$$\kappa\left(\frac{d^\kappa v_i}{dx^\kappa}\right)_0 = \left(\phi_{\kappa i} + \frac{\partial f_i}{\partial v_1}\frac{d^\kappa v_1}{dx^\kappa} + \frac{\partial f_i}{\partial v_2}\frac{d^\kappa v_2}{dx^\kappa} + \ldots + \frac{\partial f_i}{\partial v_n}\frac{d^\kappa v_n}{dx^\kappa}\right)_0; \; i = 1, 2, \ldots n, \tag{9}$$

where the $\phi^0_{\kappa i}$'s are functions of the partial differential coefficients of orders $1, 2, \ldots . \kappa$ of f_i with respect to $x, v_1, \ldots . v_n$, and of the differential coefficients of the v_i's of orders $1, 2, \ldots . \kappa - 1$, these last having been already reckoned out.

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* For a discussion of similar integrals of the equation of the 1st order see the Memoir of Briot and Bouquet already referred to. See also Poincaré, Courbes définies par une équation différentielle, Journal de Math. pure et appliquées, III, 7; IV, 1, 2.

As above, it will be assumed that the determinant

$$\Delta(x) \equiv \left| \begin{matrix} \frac{\partial f_1}{\partial v_1} - x, & \frac{\partial f_1}{\partial v_2}, & \frac{\partial f_1}{\partial v_3}, & \ldots \frac{\partial f_1}{\partial v_n} \\ \frac{\partial f_2}{\partial v_1}, & \frac{\partial f_2}{\partial v_2} - x, & \frac{\partial f_2}{\partial v_3}, & \ldots \frac{\partial f_2}{\partial v_n} \\ \frac{\partial f_3}{\partial v_1}, & \frac{\partial f_3}{\partial v_2}, & \frac{\partial f_3}{\partial v_3} - x, & \ldots \frac{\partial f_3}{\partial v_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial v_1}, & \frac{\partial f_n}{\partial v_2}, & \frac{\partial f_n}{\partial v_3}, & \ldots \frac{\partial f_n}{\partial v_n} - x \end{matrix} \right|_{x=0} \tag{10}$$

vanishes for no positive integral value of x.

Solving the equations (9) we have

$$\left(\frac{d^\kappa v_i}{dx^\kappa}\right)_0 = \phi^0_{\kappa 1} \frac{\Delta_{1i}(x)}{\Delta(x)} + \phi^0_{\kappa 2} \frac{\Delta_{2i}(x)}{\Delta(x)} + \ldots \phi^0_{\kappa n} \frac{\Delta_{ni}(x)}{\Delta(x)}; \quad i = 1, 2, \ldots n \tag{11}$$

where $\Delta_{ji}(x)$ is the minor of the element in the j^{th} row and i^{th} column of $\Delta(x)$.

Arranged with reference to the powers of x, the coefficients $\frac{\Delta_{ji}(x)}{\Delta(x)}$ have the form

$$\frac{a_{ji}x^{n-1} + \psi_{ji,1}x^{n-2} + \psi_{ji,2}x^{n-3} + \ldots \psi_{ji,n-1}}{x^n + \chi_1 x^{n-1} + \chi_2 x^{n-2} + \ldots \chi_n}$$

where $a_{ji} = 0$ when $j \gtrless i$; $= 1$ when $j = i$.

As the coefficients in this fraction are independent of x and known, and as furthermore its denominator vanishes for no value that x can take, and is of higher degree in x than its numerator, it must reach a greatest value, and that $< \infty$, for some finite value of x. Let this greatest value be C_{ji}; and let C be the modulus of the greatest of the quantities C_{ji} $(j, i = 1, 2, \ldots n)$. Then

$$\left(\frac{d^\kappa v_i}{dx^\kappa}\right)_0 \leqq C(\phi^0_{\kappa,1} + \phi^0_{\kappa,2} + \ldots + \phi^0_{\kappa,n}). \tag{12}$$

Let now r be $\leqq$ the least of the radii $r_1, r_2, \ldots r_n$, and let M be the modulus of the greatest value which any of the functions $f_1, f_2, \ldots f_n$ takes in the circle of radius ρ about $x = 0$ and the circles of radius r about $v_1 = 0$, $v_2 = 0$, etc.

Let also a be the modulus of the greatest of the differential coefficients of the first order $\left(\frac{dv_i}{dx}\right)_0$, and set $a' = a - \frac{nCM}{\rho}$.

Consider, then, the equation, algebraic in u and x,

$$u = a'x + nC\Phi,$$

where
$$\Phi = M\left(-n\frac{u}{r} - 1 + \frac{1}{\left(1-\frac{u}{r}\right)^n\left(1-\frac{x}{\rho}\right)}\right). \tag{13}$$

It may readily be shown that $\left(\frac{d^\varkappa u}{dx^\varkappa}\right)_0 > \left(\frac{d^\varkappa v_\iota}{dx^\varkappa}\right)_0$ for $\varkappa > 1$. For $\left(\frac{\partial\Phi}{\partial u}\right)_0 = 0$, so that $\left(\frac{d^\varkappa\Phi}{dx^\varkappa}\right)_0$ involves the differential coefficients $\left(\frac{du}{dx}\right)_0$, $\left(\frac{d^2u}{dx^2}\right)_0$ etc. of orders less than $\varkappa$ only and the equation

$$\left(\frac{d^\varkappa u}{dx^\varkappa}\right)_0 = nC\left(\frac{d^\varkappa\Phi}{dx^\varkappa}\right)_0$$

gives $\left(\frac{d^\varkappa u}{dx^\varkappa}\right)_0$ *explicitly* in terms of the partial differential coefficients of Φ in respect to u and x and of the differential coefficients of lower orders of u in respect to x.

Now $\left(\frac{d^\varkappa\Phi}{dx^\varkappa}\right)_0$ is greater than any of the functions $\phi^0_{\varkappa\iota}$ in equation (12). For $\phi_{\varkappa\iota} + \frac{\partial f_\iota}{\partial v_1}\frac{d^\varkappa v_1}{dx^\varkappa} + \frac{\partial f_\iota}{\partial v_2}\frac{d^\varkappa v_2}{dx^\varkappa} + \ldots\ldots + \frac{\partial f_\iota}{\partial v_n}\frac{d^\varkappa v_n}{dx^\varkappa}$ is the result of operating on f_ι $\varkappa$ times with $\frac{\partial}{\partial x} + \frac{\partial}{\partial v_1}\frac{dv_1}{dx} + \ldots\ldots + \frac{\partial}{\partial v_n}\frac{dv_n}{dx}$, while $\frac{d^\varkappa\Phi}{dx^\varkappa}$ is the result of operating on Φ $\varkappa$ times with $\frac{\partial}{\partial x} + n\frac{\partial}{\partial u}\frac{du}{dx}$.

To every differential coefficient $\frac{\partial^{n+n'+n''+\ldots\ldots} f_\iota}{\partial x^n \partial v_1^{n'} \partial v_2^{n''}\ldots\ldots}$ in $\phi_{\varkappa\iota}$ therefore corresponds the differential coefficient $\frac{\partial^{n+n'+n''+\ldots\ldots}\Phi}{\partial x^n \partial u^{n'+n''+\ldots\ldots}}$ in $\frac{d^\varkappa\Phi}{dx^\varkappa}$, involved with the same numerical coefficient; and to every $\frac{d^j v_\iota}{dx^j}$ in $\phi_{\varkappa\iota}$ $(j < \varkappa)$, corresponds $\frac{d^j u}{dx^j}$ in $\frac{d^\varkappa\Phi}{dx^\varkappa}$.

But, as is known,*

$$\text{mod}\left(\frac{\partial^{n+n'+n''+\ldots\ldots} f_\iota}{\partial x^n \partial v_1^{n'} \partial v_2^{n''}\ldots\ldots}\right)_0 < n!\ n'!\ n''!\ldots\ldots\frac{M}{\rho^n r_1^{n'} r_2^{n''}\ldots\ldots},$$

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* Vid. Briot and Bouquet, Fonc. ellip. p. 326.

while
$$\left(\frac{\partial^{n+n'+n''+\dots}\Phi}{\partial x^n \partial u^{n'+n''+\dots}}\right)_0 = n!\, n'!\, n''! \dots \frac{M}{\rho^n r_1^{n'} r_2^{n''} \dots} :$$
so that $\operatorname{mod}\left(\frac{\partial^{n+n'+n''+\dots}f_\iota}{\partial x^n \partial v_1^{n'} \partial v_2^{n''} \dots}\right)_0 < \left(\frac{\partial^{n+n'+n''+\dots}\Phi}{\partial x^n \partial u^{n'+n''+\dots}}\right)_0$.

If, therefore, $\left(\frac{d^j v_\iota}{dx^j}\right)_0 \overset{=}{<} \left(\frac{d^j u}{dx^j}\right)_0$ for $j < \kappa$, it is clear that the individual terms of $\left(\frac{d^\kappa \Phi}{dx^\kappa}\right)_0$ are greater than the corresponding terms of $\phi^0_{\kappa\iota}$, and since they are all positive, that $\left(\frac{d^\kappa \Phi}{dx^\kappa}\right)_0 > \phi^0_{\kappa\iota}$, and hence that $\left(\frac{d^\kappa u}{dx^\kappa}\right)_0$, which is equal to $nC\,\frac{d^\kappa \Phi}{dx^\kappa}$, is greater than $C(\phi^0_{\kappa 1} + \phi^0_{\kappa 2} + \dots \phi^0_{\kappa n})$; in other words (vid. 12), that $\left(\frac{d^\kappa u}{dx^\kappa}\right)_0 > \left(\frac{d^\kappa v_\iota}{dx^\kappa}\right)_0$.

But by hypothesis $\left(\frac{du}{dx}\right)_0 \geqq \left(\frac{dv_\iota}{dx}\right)_0$. It follows at once that $\left(\frac{d^2 u}{dx^2}\right)_0 > \left(\frac{d^2 v_\iota}{dx^2}\right)_0$, therefore that $\left(\frac{d^3 u}{dx^3}\right)_0 > \left(\frac{d^3 v_\iota}{dx^3}\right)_0$, ..., etc.

But u is defined as a holomorphic function of x for the neighborhood of $x = 0$ by the algebraic equation (13), or
$$\psi(u_1 x) \equiv u\left(1 - \frac{u}{r}\right)^n \left(1 - \frac{x}{\rho}\right)\left(1 + \frac{bn}{r}\right) - \left(1 - \frac{u}{r}\right)^n (a'x - b)\left(1 - \frac{x}{\rho}\right) - b = 0\,;\ (b = nCM),$$
namely within a circle about $x = 0$ whose radius is the distance to the nearest branch point or singular point of $\psi = 0$. This distance is greater than zero, being the modulus of the least root of the discriminant of $\psi = 0$ in respect to u, that is, of the equation
$$\left(1 - \frac{x}{\rho}\right)\left(1 + \frac{b - a'x}{r + bn}\right)^{n+1} = \frac{1}{1 + \frac{r}{bn}}\ \frac{(n+1)^{n+1}}{n^{n+1}}.$$
Within this circle, therefore, the series
$$u = \left(\frac{du}{dx}\right)_0 x + \left(\frac{d^2 u}{dx^2}\right)_0 \frac{x^2}{2!} + \left(\frac{d^3 u}{dx^3}\right)_0 \frac{x^3}{3!} + \dots$$
is convergent; and since its coefficients are greater than the corresponding coefficients of any of the series
$$v_\iota = \left(\frac{dv_\iota}{dx}\right)_0 x + \left(\frac{d^2 v_\iota}{dx^2}\right)_0 \frac{x^2}{2!} + \left(\frac{d^3 v_\iota}{dx^3}\right)_0 \frac{x^3}{3!} + \dots;$$
these series also converge for the same region, as was to be demonstrated.

Princeton College, March 27th, 1889.

Sur les solutions singulières des équations différentielles simultanées.

PAR E. GOURSAT.

La théorie des intégrales singulières des équations différentielles a donné lieu depuis Lagrange à un grand nombre de travaux; l'un des plus connus est sans contredit le Mémoire de Mr. Darboux publié dans le Bulletin des Sciences Mathématiques (t. IV, 1873), dont les résultats sont aujourd'hui classiques. Je me suis proposé d'étendre les théorèmes de Mr. Darboux aux équations différentielles simultanées et aux équations d'ordre supérieur. La plupart des résultats auxquels on parvient ainsi pourraient, il est vrai, se déduire des propriétés établies dans le Mémoire du même auteur *Sur les solutions singulières des équations aux dérivées partielles;*[*] mais, comme ces résultats sont indépendants de la théorie des équations aux dérivées partielles, il m'a paru intéressant de les établir directement.

1. Pour ne pas interrompre la suite des raisonnements, je vais d'abord généraliser un des théorèmes fondamentaux sur les équations différentielles du premier ordre, démontré par Briot et Bouquet dans leur célèbre Mémoire sur ce sujet.[†] Considérons un système d'équations différentielles de la forme

$$\left.\begin{aligned} x\frac{dy}{dx} &= f(x, y, z) = ax + by + cz + \ldots\ldots, \\ x\frac{dz}{dx} &= f_1(x, y, z) = a_1x + b_1y + c_1z + \ldots\ldots, \end{aligned}\right\} \qquad (1)$$

où $f(x, y, z)$ et $f_1(x, y, z)$ sont des fonctions holomorphes des variables x, y, z dans le domaine du point $x = y = z = 0$. Proposons-nous de rechercher si ces équations admettent un système d'intégrales holomorphes s'évanouissant avec x.

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[*] *Mémoires des Savants étrangers*, t. XXVII.

[†] *Journal de l'Ecole Polytechnique*, XXXVI$^{\text{ème}}$ Cahier.

Si un tel système existe, il suffira, pour avoir les développements en séries, de calculer les dérivées successives de y et de z pour $x=0$; ces dérivées s'obtiendront comme il suit. Après n différentiations successives, les équations (1) donnent

$$x\frac{d^{n+1}y}{dx^{n+1}}+n\frac{d^ny}{dx^n}=\frac{\partial f}{\partial y}\frac{d^ny}{dx^n}+\frac{\partial f}{\partial z}\frac{d^nz}{dx^n}+\ldots.,$$
$$x\frac{d^{n+1}z}{dx^{n+1}}+n\frac{d^nz}{dx^n}=\frac{\partial f_1}{\partial y}\frac{d^ny}{dx^n}+\frac{\partial f_1}{\partial z}\frac{d^nz}{dx^n}+\ldots.;$$

si on y fait $x=y=z=0$, on obtient les relations

$$\left.\begin{aligned}(n-b)\left(\frac{d^ny}{dx^n}\right)_0-c\left(\frac{d^nz}{dx^n}\right)_0\quad &=F\left[\left(\frac{dy}{dx}\right)_0,\ \left(\frac{d^2y}{dx^2}\right)_0,\ldots.\left(\frac{d^{n-1}z}{dx^{n-1}}\right)_0\right],\\ -b_1\left(\frac{d^ny}{dx^n}\right)_0+(n-c_1)\left(\frac{d^nz}{dx^n}\right)_0&=F_1\left[\left(\frac{dy}{dx}\right)_0,\ \left(\frac{d^2y}{dx^2}\right)_0,\ldots.\left(\frac{d^{n-1}z}{dx^{n-1}}\right)_0\right],\end{aligned}\right\}\quad(2)$$

F et F_1 désignant des fonctions entières des dérivées

$$\left(\frac{dy}{dx}\right)_0,\ \left(\frac{d^2y}{dx^2}\right)_0,\ldots.\left(\frac{d^{n-1}z}{dx^{n-1}}\right)_0.$$

Ces relations détermineront $\left(\frac{d^ny}{dx^n}\right)_0$ et $\left(\frac{d^nz}{dx^n}\right)_0$ au moyen des dérivées précédentes pourvu que le déterminant

$$(n-b)(n-c_1)-b_1c$$

soit différent de zéro. Par conséquent, si l'équation

$$(\omega-b)(\omega-c_1)-b_1c=0$$

n'admet pour racine aucun nombre entier positif, on pourra calculer de proche en proche toutes les dérivées successives

$$\left(\frac{dy}{dx}\right)_0,\ \left(\frac{d^2y}{dx^2}\right)_0,\ldots.\left(\frac{d^ny}{dx^n}\right)_0\ldots.,$$
$$\left(\frac{dz}{dx}\right)_0,\ \left(\frac{d^2z}{dx^2}\right)_0,\ldots.\left(\frac{d^nz}{dx^n}\right)_0\ldots.,$$

de sorte que, si les équations (1) admettent un système d'intégrales holomorphes s'évanouissant avec x, ce système est unique et les intégrales seront représentées par les développements en séries

$$y=\frac{x}{1}\left(\frac{dy}{dx}\right)_0+\frac{x^2}{1.2}\left(\frac{d^2y}{dx^2}\right)_0+\ldots.,$$
$$z=\frac{x}{1}\left(\frac{dz}{dx}\right)_0+\frac{x^2}{1.2}\left(\frac{d^2z}{dx^2}\right)_0+\ldots.$$

les coefficients étant calculés comme on vient de l'indiquer. Tout revient à démontrer la convergence de ces développements pour des valeurs de x de module suffisamment petit. Ici cette démonstration présente une difficulté spéciale, provenant de ce que les valeurs de $\left(\frac{d^ny}{dx^n}\right)_0$ et $\left(\frac{d^nz}{dx^n}\right)_0$ ne s'obtiennent pas par les seules opérations d'addition et de multiplication. En effet, en résolvant les équations (2), on aura en général dans les seconds membres des termes précédés du signe —. On évite cette difficulté en opérant de la manière suivante. Ajoutons les équations proposées (1) après les avoir multipliées respectivement par deux constantes indéterminées λ et μ; il vient

$$x\frac{d(\lambda y+\mu z)}{dx} = (\lambda a + \mu a_1)x + (\lambda b + \mu b_1)y + (\lambda c + \mu c_1)z + \dots. \tag{3}$$

Posons
$$\frac{\lambda b+\mu b_1}{\lambda} = \frac{\lambda c + \mu c_1}{\mu} = \omega,$$
ou
$$\lambda(b-\omega) + \mu b_1 = 0,$$
$$\lambda c + \mu(c_1 - \omega) = 0;$$

pour qu'on puisse satisfaire à ces relations par des valeurs de λ et μ qui ne soient pas toutes nulles, il faudra que ω soit racine de l'équation

$$(\omega - b)(\omega - c_1) - b_1 c = 0. \tag{4}$$

Soit ω_1 une racine de cette équation; on pourra satisfaire aux relations

$$\lambda(b-\omega_1) + \mu b_1 = 0,$$
$$\lambda c + \mu(c_1 - \omega_1) = 0$$

en prenant pour λ et μ des constantes dont l'une au moins ne sera pas nulle. Supposons par exemple λ différent de zéro; si on pose $\lambda y + \mu z = Y$, on pourra prendre Y et z pour inconnues à la place de y et de z et le système (1) sera remplacé par le système

$$\left.\begin{aligned} x\frac{dY}{dx} &= (\lambda a + \mu a_1)x + \omega_1 Y + \alpha x^2 + \dots, \\ x\frac{dz}{dx} &= a_1 x + \frac{b_1}{\lambda}Y + \left(c_1 - b_1\frac{\mu}{\lambda}\right)z + \alpha_1 x^2 + \dots. \end{aligned}\right\} \tag{5}$$

Remarquons encore que le coefficient de z dans la seconde équation est précisément la seconde racine ω_2 de l'équation (4). Si ω_2 est différent de ω_1, on pourra poser

$$Z = \frac{b_1}{\lambda} Y + (\omega_2 - \omega_1) z$$

et prendre Y et Z pour inconnues nouvelles ; alors les équations en Y et Z prendrons la forme

$$x \frac{dY}{dx} = a'x + \omega_1 Y + \ldots ,$$

$$x \frac{dZ}{dx} = a_1'x + \omega_2 Z + \ldots ,$$

mais cette seconde transformation, qui n'est pas toujours possible, n'est pas essentielle. Pour ne pas multiplier les notations, je suppose qu'on ait commencé par ramener le systême primitif à la forme (5); alors le théorème général qu'il s'agit de démontrer pourra s'énoncer ainsi.

Les équations

$$\left.\begin{aligned} x \frac{dy}{dx} &= ax + by + dx^2 + ey^2 + fz^2 + \ldots = f(x, y, z), \\ x \frac{dz}{dx} &= a_1 x + b_1 y + c_1 z + d_1 x^2 + e_1 y^2 + f_1 z^2 + \ldots = f_1(x, y, z), \end{aligned}\right\} \qquad (6)$$

où $f(x, y, z)$, $f_1(x, y, z)$ sont des fonctions holomorphes dans le domaine du point $x = y = z = 0$, admet un systême d'intégrales holomorphes s'évanouissant avec x pourvu qu'aucun des coefficients b, c_1 ne soit égal à un nombre entier positif.

En différentiant successivement les équations (6), on obtient

$$x \frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} + \frac{\partial f}{\partial z} \frac{dz}{dx},$$

$$x \frac{d^2z}{dx^2} + \frac{dz}{dx} = \frac{\partial f_1}{dx} + \frac{\partial f_1}{\partial y} \frac{dy}{dx} + \frac{\partial f_1}{\partial z} \frac{dz}{dx},$$

$$x \frac{d^3y}{dx^3} + 2 \frac{d^2y}{dx^2} = \frac{\partial^2 f}{dx^2} + \ldots\ldots + \frac{\partial f}{\partial y} \frac{d^2y}{dx^2} + \frac{\partial f}{\partial z} \frac{d^2z}{dx^2},$$

$$x \frac{d^3z}{dx^3} + 2 \frac{d^2z}{dx^2} = \frac{\partial^2 f_1}{dx^2} + \ldots\ldots + \frac{\partial f_1}{\partial y} \frac{d^2y}{dx^2} + \frac{\partial f_1}{\partial z} \frac{d^2z}{dx^2},$$

. . . .

ce qui donne, en faisant $x=y=z=0$,

$$\begin{aligned}
&\left(\frac{dy}{dx}\right)_0(1-b)=a,\\
&\left(\frac{dz}{dx}\right)_0(1-c_1)=a_1+b_1\left(\frac{dy}{dx}\right)_0,\\
&\left(\frac{d^2y}{dx^2}\right)_0(2-b)=\left(\frac{\partial^2 f}{\partial x^2}\right)_0+\cdots\cdot+2\left(\frac{\partial^2 f}{\partial y\partial z}\right)_0\left(\frac{dy}{dx}\right)_0\left(\frac{dz}{dx}\right)_0,\\
&\left(\frac{d^2z}{dx^2}\right)_0(2-c_1)=\left(\frac{\partial^2 f_1}{\partial x^2}\right)_0+\cdots\cdot+2\left(\frac{\partial^2 f_1}{\partial y\partial z}\right)_0\left(\frac{dy}{dx}\right)_0\left(\frac{dz}{dx}\right)_0+b_1\left(\frac{d^2y}{dx^2}\right)_0,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

On en déduit

$$\left.\begin{aligned}
&\left(\frac{dy}{dx}\right)_0=\frac{a}{1-b},\quad \left(\frac{dz}{dx}\right)_0=\frac{a_1+b_1\left(\frac{dy}{dx}\right)_0}{1-c_1},\\
&\left(\frac{d^2y}{dx^2}\right)_0=\frac{\left(\frac{\partial^2 f}{\partial x^2}\right)_0+\cdots\cdot+2\left(\frac{\partial^2 f}{\partial y\partial z}\right)_0\left(\frac{dy}{dx}\right)_0\left(\frac{dz}{dx}\right)_0}{2-b},\\
&\left(\frac{d^2z}{dx^2}\right)_0=\frac{\left(\frac{\partial^2 f_1}{\partial x^2}\right)_0+\cdots\cdot+2\left(\frac{\partial^2 f_1}{\partial y\partial z}\right)_0\left(\frac{dy}{dx}\right)_0\left(\frac{dz}{dx}\right)_0+b_1\left(\frac{d^2y}{dx^2}\right)_0}{2-c_1},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}\right\}\qquad(7)$$

Pour démontrer la convergence des développements ainsi obtenus, il suffit d'employer un artifice tout-à-fait analogue à celui par lequel Briot et Bouquet ont démontré le théorème correspondant dans le cas d'une seule équation. Supposons que les fonctions $f(x,y,z)$ et $f_1(x,y,z)$ soient développables en séries convergentes pour tous les systêmes de valeurs de x, y, z de module inférieur à r, et soit M une limite supérieure du module de ces fonctions dans ce domaine. Posons

$$\begin{aligned}
\phi(x,u,v)&=Ax+Bu+\frac{M}{\left(1-\frac{x}{r}\right)\left(1-\frac{u}{r}\right)\left(1-\frac{v}{r}\right)}-M\left\{1+\frac{x+u+v}{r}\right\}\\
&=Ax+Bu+M\left\{\frac{x^2+u^2+v^2+xu+xv+uv}{r^2}+\cdots\cdot\right\},\\
\phi_1(x,u,v)&=A_1x+B_1u+C_1v+M\left\{\frac{x^2+u^2+v^2+xu+xv+uv}{r^2}+\cdots\right\};
\end{aligned}$$

toutes des dérivées partielles des fonctions ϕ et ϕ_1, à partir des secondes, sont réelles et positives pour $x=u=v=0$ et sont supérieures aux modules des dérivées correspondantes des fonctions f et f_1 pour $x=y=z=0$. Nous prendrons pour

A, A_1, B_1 les modules de a, a_1, b_1 respectivement et pour B, C_1 des nombres positifs inférieurs à l'unité qui seront déterminés plus loin d'une façon plus précise.

Le système d'équations

$$\left.\begin{aligned} u-\phi\,(x,u,v)&=0\\ v-\phi_1(x,u,v)&=0 \end{aligned}\right\} \tag{8}$$

définit un système de fonctions holomorphes s'évanouissant avec x, car le déterminant fonctionnel des premiers membres se réduit à $(1-B)(1-C_1)$ pour $x=u=v=0$. Dans un certain domaine autour de l'origine, ces fonctions seront représentées par des développements en séries convergentes

$$\left.\begin{aligned} u&=\frac{x}{1}\left(\frac{du}{dx}\right)_0+\frac{x^2}{1.2}\left(\frac{d^2u}{dx^2}\right)_0+\ldots.,\\ v&=\frac{x}{1}\left(\frac{dv}{dx}\right)_0+\frac{x^2}{1.2}\left(\frac{d^2v}{dx^2}\right)_0+\ldots. \end{aligned}\right\} \tag{9}$$

dont on pourra calculer tous les coefficients au moyen des équations (8). En différentiant ces relations on a, en effet,

$$\begin{aligned} \frac{du}{dx}&=\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial u}\,\frac{du}{dx}+\frac{\partial\phi}{\partial v}\,\frac{dv}{dx},\\ \frac{dv}{dx}&=\frac{\partial\phi_1}{\partial x}+\frac{\partial\phi_1}{\partial u}\,\frac{du}{dx}+\frac{\partial\phi_1}{\partial v}\,\frac{dv}{dx},\\ \frac{d^2u}{dx^2}&=\frac{\partial^2\phi}{\partial x^2}+\ldots\ldots+\frac{\partial\phi}{\partial u}\,\frac{d^2u}{dx^2}+\frac{\partial\phi}{\partial v}\,\frac{d^2v}{dx^2},\\ \frac{d^2v}{dx^2}&=\frac{\partial^2\phi_1}{\partial x^2}+\ldots\ldots+\frac{\partial\phi_1}{\partial u}\,\frac{d^2u}{dx^2}+\frac{\partial\phi_1}{\partial v}\,\frac{d^2v}{dx^2},\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

On en déduit, en faisant $x=u=v=0$,

$$\left.\begin{aligned} &\left(\frac{du}{dx}\right)_0=\frac{A}{1-B},\quad \left(\frac{dv}{dx}\right)_0=\frac{A_1+B_1\left(\frac{du}{dx}\right)_0}{1-C_1},\\ &\left(\frac{d^2u}{dx^2}\right)_0=\frac{\left(\frac{\partial^2\phi}{\partial x^2}\right)_0+\ldots.+2\left(\frac{\partial^2\phi}{\partial u\partial v}\right)_0\left(\frac{du}{dx}\right)_0\left(\frac{dv}{dx}\right)_0}{1-B},\\ &\left(\frac{d^2v}{dx^2}\right)_0=\frac{\left(\frac{\partial^2\phi_1}{\partial x^2}\right)_0+\ldots.+2\left(\frac{\partial^2\phi_1}{\partial u\partial v}\right)_0\left(\frac{du}{dx}\right)_0\left(\frac{dv}{dx}\right)_0+B_1\left(\frac{\partial^2u}{\partial x^2}\right)_0}{1-C_1},\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{10}$$

On voit que tous ces coefficients sont réels et positifs. Puisqu'aucun des coefficients b, c_1 n'est un nombre entier, le module de $n - b$, où n est un nombre entier positif, reste supérieur à une certaine limite m et, en prenant le nombre B tel que $1 - B$ soit inférieur à m, on aura constamment

$$1 - B < |n - b|, \text{ ou } \frac{1}{1-B} > \left|\frac{1}{n-b}\right|;$$

on choisira de même le nombre positif C_1 de façon que l'on ait, pour toute valeur entière et positive de n,

$$1 - C_1 < |n - c_1|.$$

Cela posé, la comparaison des formules (7) et (10) nous montre que l'on aura

$$\left|\frac{dy}{dx}\right|_0 < \left(\frac{du}{dx}\right)_0, \quad \left|\frac{dz}{dx}\right|_0 < \left(\frac{dv}{dx}\right)_0,$$
$$\left|\frac{d^2y}{dx^2}\right|_0 < \left(\frac{d^2u}{dx^2}\right)_0, \quad \left|\frac{d^2z}{dx^2}\right|_0 < \left(\frac{d^2v}{dx^2}\right)_0$$

et, d'une manière générale,

$$\left|\frac{d^ny}{dx^n}\right|_0 < \left(\frac{d^nu}{dx^n}\right)_0, \quad \left|\frac{d^nz}{dx^n}\right|_0 < \left(\frac{d^nv}{dx^n}\right)_0.$$

Les séries (9) étant convergentes dans un certain domaine autour de l'origine, il en sera de même *a fortiori* des séries

$$\frac{x}{1}\left(\frac{dy}{dx}\right)_0 + \frac{x^2}{1.2}\left(\frac{d^2y}{dx^2}\right)_0 + \ldots\ldots,$$
$$\frac{x}{1}\left(\frac{dz}{dx}\right)_0 + \frac{z^2}{1.2}\left(\frac{d^2z}{dx^2}\right)_0 + \ldots\ldots,$$

ce qui démontre le théorème.

Si nous revenons à la forme primitive des équations (1), nous pouvons énoncer la proposition générale suivante :

Les équations (1) *admettent un système d'intégrales holomorphes s'évanouissant avec x pourvu que l'équation*

$$(\omega - b)(\omega - c_1) - b_1 c = 0$$

n'admette pour racine aucun nombre entier positif. Un cas particulier de ce théorème a été démontré par Mr. Picard (Comptes-rendus, 1878).

(2). Nous allons maintenant étudier le cas où l'équation précédente admet pour racine un nombre entier positif; plusieurs cas sont à distinguer suivant qu'elle admet pour racines ou deux nombres entiers positifs.

Supposons d'abord qu'il n'y ait qu'un seul nombre entier positif racine de l'équation

$$(\omega - b)(\omega - c_1) - b_1 c = 0;$$

soit p cette racine et soit ω_2 la seconde racine. On pourra, comme on l'a vu plus haut, ramener le système d'équations proposées à la forme simple

$$x\frac{dy}{dx} = ax + py + dx^2 + \ldots.,$$
$$x\frac{dz}{dx} = a_1x + \omega_2 z + d_1x^2 + \ldots.$$

On diminue les coefficients p et ω_2 d'une unité en posant

$$y = x\left(\frac{a}{1-p} + Y\right), \quad z = x\left(\frac{a_1}{1-\omega_2} + Z\right);$$

après $p - 1$ transformations de cette espèce, on sera ramené à un système d'équations de la forme suivante

$$\left.\begin{aligned} x\frac{dy}{dx} &= ax + y + dx^2 + \ldots. \\ x\frac{dz}{dx} &= a_1x + c_1z + d_1x^2 + \ldots. \end{aligned}\right\} \qquad (11)$$

où $c_1 = \omega_2 - p - 1$ n'est pas un nombre entier positif.

En général ce système n'admet pas d'intégrales holomorphes s'evanouissant avec x; en effet, soit

$$y = A_0x + A_1x^2 + \ldots.,$$
$$z = B_0x + B_1x^2 + \ldots.$$

les développements en séries de ces intégrales. En substituant dans les équations (11), on aurait

$$x(A_0 + 2A_1x + \ldots.) = ax + (A_0x + A_1x^2 + \ldots.) + x^2(\ldots.)$$

et par suite

$$A_0 = A_0 + a;$$

ce qui exige que le coefficient a soit nul. Ainsi, *lorsque le coefficient a est différent de zéro, les équations* (11) *n'admettent pas d'intégrales holomorphes s'évanouissant avec x.*

Supposons maintenant que a soit nul; si on pose

$$y=\lambda x, \quad z=\mu x,$$

les équations (11) deviennent

$$x\frac{d\lambda}{dx}=x\{\alpha+\beta\lambda+\gamma\mu+\dots\},$$
$$x\frac{d\mu}{dx}=a_1+\mu(c_1-1)+x\{\alpha_1+\beta_1\lambda+\gamma_1\mu+\dots\};$$

la valeur initiale μ_0 de μ pour $x=0$ doit vérifier la relation $a_1+\mu_0(c_1-1)=0$, tandis que la valeur initiale λ_0 de λ peut être choisie arbitrairement. Si on prend pour λ_0 une constante quelconque et si on pose

$$\lambda=\lambda_0+\lambda', \quad \mu=\frac{a_1}{1-c_1}+\mu',$$

on aboutit au nouveau système

$$x\frac{d\lambda'}{dx}=x\{\alpha'+\beta\lambda'+\gamma\mu'+\dots\},$$
$$x\frac{d\mu'}{dx}=(c_1-1)\mu'+x\{\alpha_1'+\beta_1\lambda'+\gamma_1\mu'+\dots\}$$

auquel on peut appliquer le théorème général démontré plus haut. Ainsi, *lorsque le coefficient a est nul, les équations* (11) *admettent une infinité d'intégrales holomorphes s'évanouissant avec x; ces intégrales dépendent d'une constante arbitraire λ_0.*

Supposons en second lieu que l'équation

$$(\omega-b)(\omega-c_1)-b_1c=0$$

admette pour racines deux nombres entiers positifs p et $p+q$ $(q\geqq 0)$. Par une série de transformations de même nature que les précédentes, on ramènera le système proposé à la forme

$$\left.\begin{aligned} x\frac{dy}{dx}&=ax+y+dx^2+\dots,\\ x\frac{dz}{dx}&=a_1x+b_1y+qz+d_1x^2+\dots \end{aligned}\right\} \qquad (12)$$

le coefficient b_1 pouvant être pris égal à zéro lorsque q est supérieur à l'unité. On démontre, comme tout-à-l'heure que ce système ne peut admettre d'intégrales holomorphes s'évanouissant avec x que si a est nul.

Si on a en même temps $q=1$, le système (12) devient

$$x\frac{dy}{dx}=y+dx^2+\dots\,,$$
$$x\frac{dz}{dx}=a_1x+b_1y+z+d_1x^2+\dots\,.$$

Il peut arriver que l'on ait aussi $b_1=0$; dans ce cas, en raisonnant comme plus haut, on démontrera que l'on doit avoir aussi $a_1=0$, et, si cette condition est satisfaite, on aura une infinité d'intégrales holomorphes s'évanouissant avec x car, en posant $y=\lambda x$, $z=\mu x$, on sera ramené à un système d'équations

$$\frac{d\lambda}{dx}=\phi\,(x,\lambda,\mu),$$
$$\frac{d\mu}{dx}=\phi_1(x,\lambda,\mu),$$

où les seconds membres sont des fonctions holomorphes de x, de λ et de μ et on pourra choisir arbitrairement les valeurs initiales λ_0, μ_0 des intégrales de ce système.

Si b_1 n'est pas nul, le changement de variables

$$y=\lambda x,\quad z=\mu x$$

conduit au nouveau système

$$x\frac{d\lambda}{dx}=x(\alpha+\beta\lambda+\gamma\mu+\dots),$$
$$x\frac{d\mu}{dx}=a_1+b_1\lambda+x(\alpha_1+\beta_1\lambda+\gamma_1\mu+\dots).$$

La valeur initiale λ_0 de λ sera donnée par la relation

$$a_1+b_1\lambda_0=0,$$

mais la valeur initiale μ_0 de μ reste arbitraire. En posant ensuite

$$\lambda=-\frac{a_1}{b_1}+\lambda',\quad \mu=\mu_0+\mu',$$

on arrive au système

$$x\frac{d\lambda'}{dx}=x(\alpha'+\beta'\lambda'+\gamma'\mu'+\dots),$$
$$x\frac{d\mu'}{dx}=b_1\lambda'+x(\alpha_1'+\beta_1'\lambda'+\gamma_1'\mu'+\dots)$$

auquel on peut appliquer le théorème général. Il existe donc une infinité d'intégrales holomorphes dépendant d'une constante arbitraire μ_0.

Il reste à examiner le cas où q est un nombre entier supérieur à l'unité; on peut se borner, comme on l'a vu, à considérer le systême

$$\left.\begin{aligned} x\frac{dy}{dx} &= y + dx^2 + \ldots\ldots, \\ x\frac{dz}{dx} &= a_1x + qz + d_1x^2 + \ldots\ldots \end{aligned}\right\} \qquad (12')$$

Si on pose $y = x(\lambda_0 + \lambda)$, $z = x\left(\frac{a}{1-q} + \mu\right)$, λ_0 désignant une constante indéterminée, on sera conduit à un systême

$$x\frac{d\lambda}{dx} = x\phi(x, \lambda, \mu), \quad x\frac{d\mu}{dx} = a_2x + (q-1)\mu + d_2x^2 + \ldots\ldots$$

Pour ce nouveau systême, les deux racines de l'équation en ω sont 0 et $q-1$; si q est plus grand que 2, en diminuant successivement ces racines par le procédé qui a été expliqué plus haut de $q-2$ unités on sera ramené finalement à un systême de la forme

$$\begin{aligned} x\frac{d\lambda^{(n)}}{dx} &= a^{(n)}x + (2-q)\lambda^{(n)} + \ldots\ldots, \\ x\frac{d\mu^{(n)}}{dx} &= a_1^{(n)}x + \mu^{(n)} + \ldots\ldots \end{aligned}$$

et ce systême admettra des intégrales holomorphes s'évanouissant avec x pourvu que $a_1^{(n)}$ soit nul. Comme ce coefficient dépendra en général de λ_0, on voit qu'on aura une équation pour déterminer les valeurs convenables de λ_0. Ayant pris pour λ_0 une racine de cette équation, on aura une infinité d'intégrales holomorphes dépendant d'une constante arbitraire $\mu_0^{(n)}$.

3. Le théorème général qui vient d'être démontré s'étend sans difficulté à des systêmes formés d'un nombre quelconque d'équations de la forme suivante :

$$\begin{aligned} x\frac{dy_1}{dx} &= a_1x + b_1y_1 + c_1y_2 + \ldots\ldots + l_1y_n + p_1x^2 + \ldots\ldots, \\ x\frac{dy_2}{dx} &= a_2x + b_2y_1 + c_2y_2 + \ldots\ldots + l_2y_n + p_2x^2 + \ldots\ldots, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x\frac{dy_n}{dx} &= a_nx + b_ny_1 + c_ny_2 + \ldots\ldots + l_ny_n + p_nx^2 + \ldots\ldots \end{aligned}$$

Un pareil système d'équations admet toujours un système unique d'intégrales holomorphes s'évanouissant avec x, pourvu qu'aucune des racines de l'équation

$$\begin{vmatrix} b_1-\omega & c_1 & \ldots\ldots & l_1 \\ b_2 & c_2-\omega & \ldots\ldots & l_2 \\ \ldots & \ldots & \ldots & \ldots \\ b_n & c_n & \ldots\ldots & l_n-\omega \end{vmatrix} = 0$$

ne soit égale à un nombre entier positif. Si $\omega_1, \omega_2, \ldots\ldots \omega_n$ sont les n racines de cette équation, on commencera par ramener le système proposé à la forme

$$\begin{aligned} x\frac{dy_1}{dx} &= a_1x + \omega_1y_1 + \ldots\ldots + p_1x^2 + \ldots\ldots, \\ x\frac{dy_2}{dx} &= a_2x + b_2y_1 + \omega_2y_2 + \ldots\ldots + p_2x^2 + \ldots\ldots, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x\frac{dy_n}{dx} &= a_nx + b_ny_1 + c_ny_2 + \ldots\ldots + \omega_ny_n + p_nx^2 + \ldots\ldots \end{aligned}$$

et la démonstration s'achèvera comme dans le cas de deux équations. On discutera de la même façon le cas où l'équation en ω admet pour racines un ou plusieurs nombres entiers positifs.

On peut ramener à la forme précédente certains systèmes d'équations simultanées qui se présentent sous forme indéterminée. Considérons, pour fixer les idées, le système de trois équations

$$\frac{dx}{\Sigma Ax^{\alpha}y^{\beta}z^{\gamma}} = \frac{dy}{\Sigma A'x^{\alpha'}y^{\beta'}z^{\gamma'}} = \frac{dz}{\Sigma A''x^{\alpha''}y^{\beta''}z^{\gamma''}}$$

où les trois dénominateurs sont nuls pour $x=y=z=0$. Pour avoir les intégrales de ce système qui sont nulles pour $x=0$, cherchons d'abord le degré infinitésimal de y et de z par rapport à x. Soit λ le degré de y et μ le degré de z; supposons que les termes écrits au dénominateur soient les termes de moindre degré en x après qu'on a fait la substitution. Il faudra que l'on ait

$$\alpha + \lambda\beta + \mu\gamma = \alpha' + \lambda\beta' + \mu\gamma' - \lambda + 1 = \alpha'' + \lambda\beta'' + \mu\gamma'' - \mu + 1,$$

ou

$$\alpha + \lambda(\beta+1) + \mu(\gamma+1) = \alpha' + 1 + \lambda\beta' + \mu(\gamma'+1) = \alpha'' + 1 + \lambda(\beta''+1) + \gamma''.$$

Imaginons que l'on ait marqué les points de l'espace de coordonnées

$$(\alpha,\ \beta+1,\ \gamma+1),\ (\alpha'+1,\ \beta',\ \gamma'+1),\ (\alpha''+1,\ \beta''+1,\ \gamma'');$$

les relations précédentes expriment qu'il existe un plan

$$X + \lambda Y + \mu Z = p$$

passant par trois de ces points au moins et laissant tous les autres points et l'origine de côtés différents. La détermination de ces plans s'effectuera sans difficulté par une construction géométrique tout-à-fait analogue à la construction plane bien connue pour développer les racines d'une équation algébrique. Supposons que l'on ait trouvé des valeurs de λ et de μ répondant à la question ; ces valeurs seront commensurables et on pourra poser

$$\lambda = \frac{p}{r}, \quad \mu = \frac{q}{r},$$

p, q, r étant trois nombres entiers positifs sans facteur commun. Si dans les équations proposées on fait le changement de variables

$$x = t^r, \quad y = t^p u, \quad z = t^q v,$$

elles deviennent, après supression d'un facteur commun,

$$t\frac{du}{dt} = \frac{r(A'u^{\beta'}v^{\gamma'} + \ldots .) - pu(Au^{\beta}v^{\gamma} + \ldots .)}{Au^{\beta}v^{\gamma} + \ldots .},$$
$$t\frac{dv}{dt} = \frac{r(A''u^{\beta''}v^{\gamma''} + \ldots .) - qv(Au^{\beta}v^{\gamma} + \ldots .)}{Au^{\beta}v^{\gamma} + \ldots .}.$$

Les valeurs initiales u_0, v_0 de u et v pour $t = 0$ sont déterminées par les équations simultanées

$$r(A'u_0^{\beta'}v_0^{\gamma'} + \ldots .) - pu_0(Au_0^{\beta}v_0^{\gamma} + \ldots .),$$
$$r(A''u_0^{\beta''}v_0^{\gamma''} + \ldots .) - qv_0(Au_0^{\beta}v_0^{\gamma} + \ldots .).$$

Soient u_0, v_0 une solution de ce système ; en posant

$$u = u_0 + U, \quad v = v_0 + V,$$

on sera ramené à un système de la forme (1), pourvu que l'on n'ait pas en même temps

$$Au_0^{\beta}v_0^{\gamma} + \ldots . = 0;$$

dans ce cas, on serait conduit à effectuer une nouvelle transformation de même espèce. On opérera de même dans le cas d'un nombre quelconque d'équations.

4. L'étude des intégrales non holomorphes du système (1) serait sans doute très-intéressante ; mais, comme cette étude n'est pas essentielle pour la suite, je la laisse de côté dans ce travail. Je m'occuperai seulement du cas particulier où

les racines de l'équation en ω ont toutes les deux leur partie réelle négative. Il importe de définir exactement ce qu'on doit entendre par intégrales s'évanouissant avec x. Je suppose que la variable x tend vers l'origine suivant un chemin de longueur finie ayant à l'origine une tangente déterminée, de façon que l'argument de x reste fini. En outre, je ne considère que des intégrales d'un degré infinitésimal déterminé par rapport à x, c'est-à-dire qui peuvent se mettre sous la forme $x^\lambda(\varkappa+\varepsilon)$, λ ayant sa partie réelle positive, $\varkappa$ étant une constante différente de zéro et ε une fonction de x qui tend vers zéro dans les mêmes conditions que la première. *Avec ces restrictions, le système* (1) *n'admet pas d'autres intégrales s'évanouissant avec x que les intégrales holomorphes lorsque les parties réelles des racines de l'équation en ω sont négatives.*

Soient ω_1, ω_2 les deux racines de l'équation en ω que je suppose distinctes pour fixer les idées; le système (1) pourra se ramener à la forme

$$x\frac{dy}{dx}=ax+\omega_1 y+dx^2+\ldots\ldots,$$
$$x\frac{dz}{dx}=a_1x+\omega_2 z+d_1x^2+\ldots\ldots$$

Soient $y=y_1$, $z=z_1$ les intégrales holomorphes de ce système. En posant $y=y_1+u$, $z=z_1+v$ et en remplaçant y_1 et z_1 par leurs développements en séries on est conduit aux équations

$$x\frac{du}{dx}=\omega_1 u+u\phi\ (x,u,v)+v\psi\ (x,u,v),$$
$$x\frac{dv}{dx}=\omega_2 v\ +u\phi_1(x,u,v)+v\psi_1(x,u,v)$$

où ϕ, ϕ_1, ψ, ψ_1 sont des séries ordonnées suivant les puissances croissantes de x, u, v sans termes constants. Je dis qu'un pareil système ne peut admettre d'intégrales de la forme

$$u=x^\lambda(\varkappa+\varepsilon),\quad v=x^\mu(\varkappa'+\varepsilon'),$$

les parties réelles de λ et de μ étant positives. En effet, supposons que la partie réelle de μ soit égale ou supérieure à la partie réelle de λ. En substituant les valeurs précédentes de u et de v dans la première des équations, on aura l'égalité

$$\lambda x^\lambda(\varkappa+\varepsilon)+x^{\lambda+1}\frac{d\varepsilon}{dx}=\omega_1x^\lambda(\varkappa+\varepsilon)+x^\lambda(\varkappa+\varepsilon)\phi(x,u,v)+x^\mu(\varkappa'+\varepsilon')\psi(x,u,v),$$

ou, en divisant par x^λ,

$$(x+\varepsilon)(\lambda-\omega_1)=-x\frac{d\varepsilon}{dx}+(x+\varepsilon)\varphi(x,u,v)+x^{\mu-\lambda}(x'+\varepsilon')\psi(x,u,v).$$

Imaginons maintenant que x tende vers zéro suivant un chemin de longueur finie ayant une tangente à l'origine ; les deux termes

$$(x+\varepsilon)\varphi(x,u,v),\quad x^{\mu-\lambda}(x'+\varepsilon')\psi(x,u,v)$$

tendent vers zéro. Il doit en être de même du produit $x\frac{d\varepsilon}{dx}$, au moins pour une certaine loi de décroissement de x. Supposons en effet que le module de $x\frac{d\varepsilon}{dx}=\pi(x)$ reste constamment supérieur à une certaine limite m. De la relation

$$x\frac{d\varepsilon}{dx}=\pi(x)$$

on tire

$$\frac{dx}{x}=\frac{d\varepsilon}{\pi(x)}$$

et

$$L\left(\frac{x_1}{x_0}\right)=\int_{x_0}^{x_1}\frac{d\varepsilon}{\pi(x)};$$

lorsque x_0 tend vers zéro, le module du premier membre augmente indéfiniment tandis que le second conserverait une valeur finie ; ce qui est impossible. Le premier membre de la relation précédente doit donc tendre vers zéro avec x, ce qui ne peut avoir lieu que si $\lambda=\omega_1$. Mais alors la fonction $x^\lambda(x+\varepsilon)$ ne tendrait plus vers zéro avec x puisque la partie réelle de λ serait négative. On raisonnerait de même dans le cas où l'équation en ω aurait ses racines égales.

Il est à remarquer que le théorème n'est plus vrai si une seule des racines de l'équation en ω a sa partie réelle négative. Par exemple, le système

$$x\frac{du}{dx}=\frac{u}{2},$$

$$x\frac{dv}{dx}=-v+\frac{5}{2}u^3$$

admet l'intégrale

$$u=x^{\frac{1}{2}},\quad v=x^{\frac{3}{2}}.$$

II.

5. Considérons une *congruence* de courbes, définie par les équations

$$\left.\begin{aligned}f(x,y,z,a,b)&=0,\\ \varphi(x,y,z,a,b)&=0\end{aligned}\right\}\qquad(13)$$

où a et b désignent deux constantes arbitraires. Par chaque point de l'espace il passe en général un nombre fini de ces courbes; car, si l'on remplace dans les équations (13) x, y, z par les coordonnées x_0, y_0, z_0 de ce point, on a deux équations pour déterminer a et b. Si l'on joint aux équations (13) la nouvelle équation

$$\frac{\partial f}{\partial a}\frac{\partial \phi}{\partial b}-\frac{\partial f}{\partial b}\frac{\partial \phi}{\partial a}=\frac{D(f,\phi)}{D(a,b)}=0, \tag{14}$$

on détermine sur chaque courbe de la congruence un certain nombre de points appelés *points focaux;* le lieu de ces points focaux, quand on considère toutes les courbes de la congruence, forme en général une surface appelée *surface focale*, dont on obtiendra l'équation en éliminant a et b entre les équations (13) et (14) et qui se compose d'autant de nappes qu'il y a de points focaux sur chaque courbe de la congruence. Les courbes de la congruence sont tangentes en chacun de leurs points focaux aux différentes nappes de la surface focale. De plus, si l'on veut assembler les courbes de la congruence de façon à ce qu'elles aient une enveloppe, cette enveloppe sera située sur la surface focale et il est aisé de voir qu'elle sera définie par une équation différentielle du premier ordre. En effet, en chaque point A de la surface focale on connaît la direction de la tangente à la courbe de la congruence tangente à la surface en ce point et par suite la direction de la tangente à la courbe enveloppe qui passe par ce point.*

Regardons maintenant, dans les équations (13), y et z comme des fonctions de la variable indépendante x définies par ces équations, a et b ayant des valeurs constantes quelconques. Une différentiation par rapport à x nous donne les deux nouvelles relations

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}y'+\frac{\partial f}{\partial z}z'=0,$$

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}y'+\frac{\partial \phi}{\partial z}z'=0,$$

où

$$y'=\frac{dy}{dx}, \quad z'=\frac{dz}{dx}.$$

En éliminant a et b entre ces équations et les équations (13) on est conduit à un système d'équations différentielles de la forme

$$\left.\begin{array}{l}F(x,y,z,y',z')=0,\\ \Phi(x,y,z,y',z')=0;\end{array}\right\} \tag{15}$$

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* Pour la démonstration de ces propriétés, voir le tome II de la *Théorie générale des surfaces* de Mr. Darboux. Livre IV, Chapitre 1er.

ces équations sont vérifiées par les fonctions y et z de x définies par les équations (13), quelles que soient les valeurs constantes attribuées aux paramètres a et b. Nous dirons pour abréger que les courbes de la congruence sont les courbes intégrales du système (15), et forment l'intégrale générale. Mais ces équations (15) admettent en outre une infinité d'autres intégrales non comprises dans l'intégrale générale ; en effet, puisque ces équations établissent une relation entre un point de l'espace et la tangente à la courbe intégrale qui passe par ce point, il est clair que toute courbe tangente en chacun de ces points à une courbe intégrale sera elle-même une courbe intégrale. Par suite, toutes les courbes enveloppes des courbes de la congruence sont aussi des intégrales du système (15), et il est clair qu'en général elles ne font pas partie des courbes de la congruence ; nous les appellerons *intégrales singulières*. Nous sommes donc conduits à la proposition suivante :

Les équations différentielles (15) *admettent une infinité d'intégrales singulières, qui sont définies par une équation différentielle du premier ordre.*

6. Voici comment on pourra obtenir la surface focale et les solutions singulières en partant des équations différentielles elles-mêmes. Une des propriétés caractéristiques de la surface focale est la suivante ; par chaque point de cette surface passent deux courbes intégrales tangentes l'une à l'autre, une courbe de la congruence et une courbe enveloppe située sur la surface focale. Soient x_0, y_0, z_0 les coordonnées d'un point de cette surface et y'_0, z'_0 les valeurs correspondantes de y' et de z', relatives à ces deux intégrales. Le déterminant fonctionnel

$$\frac{D(F, \Phi)}{D(y', z')}$$

devra être nul pour $x = x_0$, $y = y_0$, $z = z_0$, $y' = y'_0$, $z' = z'_0$. En effet, si ce déterminant était différent de zéro, on tirerait des équations (15)

$$\begin{aligned} y' &= y'_0 + P(x - x_0, \ y - y_0, \ z - z_0), \\ z' &= z'_0 + Q(x - x_0, \ y - y_0, \ z - z_0), \end{aligned}$$

P et Q désignant des séries convergentes ordonnées suivant les puissances croissantes de $x - x_0$, $y - y_0$, $z - z_0$, s'évanouissant pour $x = x_0$, $y = y_0$, $z = z_0$. D'après le théorème fondamental de la théorie des équations différentielles, on en déduirait pour y et z des séries convergentes ordonnées suivant les puissances croissantes de $x - x_0$,

$$\begin{aligned} y &= y_0 + y'_0(x - x_0) + \alpha \ (x - x_0)^2 + \dots, \\ z &= z_0 + z'_0(x - x_0) + \alpha_1(x - x_0)^2 + \dots; \end{aligned}$$

il n'y aurait donc qu'une seule courbe intégrale tangente au point (x_0, y_0, z_0) à la droite
$$\frac{X-x_0}{1}=\frac{Y-y_0}{y_0'}=\frac{Z-z_0}{z_0'}.$$

Par conséquent, il faudra que pour tout point de la surface focale les équations (15) et l'équation
$$\frac{D(F,\Phi)}{D(y',z')}=0 \tag{16}$$
admettent une solution commune en y', z'. Ainsi, *on obtiendra l'équation de la surface focale en éliminant* y' *et* z' *entre les équations* (15) *et* (16).

Soit $P(x, y, z)=0$ l'équation ainsi obtenue; les solutions singulières étant situées sur la surface focale, en éliminant z et z' entre les équations (15) et l'équation de cette surface, on aura une certaine équation différentielle du premier ordre
$$\psi(x, y, y')=0$$
qui définit les projections des solutions singulières sur le plan des xy.

Si on étudie de plus près le procédé précédent, il donne lieu à plusieurs remarques. Des équations (15) on déduit, en différentiant,
$$\left.\begin{aligned}\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}y'+\frac{\partial F}{\partial z}z'+\frac{\partial F}{\partial y'}y''+\frac{\partial F}{\partial z'}z''=0,\\ \frac{\partial \Phi}{\partial x}+\frac{\partial \Phi}{\partial y}y'+\frac{\partial \Phi}{\partial z}z'+\frac{\partial \Phi}{\partial y'}y''+\frac{\partial \Phi}{\partial z'}z''=0;\end{aligned}\right\} \tag{17}$$
si on considère en particulier une intégrale singulière pour laquelle
$$\frac{\partial F}{\partial y'}\frac{\partial \Phi}{\partial z'}-\frac{\partial F}{\partial z'}\frac{\partial \Phi}{\partial y'}=0,$$
on pourra éliminer y'' et z'' entre les deux équations (17) et on obtient ainsi les deux nouvelles relations
$$\left.\begin{aligned}\frac{D(F,\Phi)}{D(x,y')}+\frac{D(F,\Phi)}{D(y,y')}y'+\frac{D(F,\Phi)}{D(z,y')}z'=0,\\ \frac{D(F,\Phi)}{D(x,z')}+\frac{D(F,\Phi)}{D(y,z')}y'+\frac{D(F,\Phi)}{D(z,z')}z'=0.\end{aligned}\right\} \tag{18}$$

Les équations (16) et (18) se réduisent à deux équations distinctes; néanmoins, pour plus de symétrie, je les conserverai toutes les trois. On voit donc que pour tout point de la surface $P(x, y, z) = 0$ les équations (15), (16) et (18) doivent admettre une solution commune en y', z'. Or il est clair que, si les fonctions F et Φ sont quelconques, il n'existera pas de surface jouissant de cette propriété, et par conséquent il ne pourra exister d'intégrales singulières.

On peut encore s'en rendre compte comme il suit. La première des équations (15) exprime que la tangente à toute courbe intégrale passant en un point de coordonnées x, y, z est située sur un cône T ayant pour équation

$$F\left(x, y, z, \frac{Y-y}{X-x}, \frac{Z-z}{X-x}\right) = 0;$$

de même, la seconde équation (15) exprime que cette tangente est située sur un second cône T' ayant pour équation

$$\Phi\left(x, y, z, \frac{Y-y}{X-x}, \frac{Z-z}{X-x}\right) = 0.$$

En général, ces deux cônes T, T' se coupent suivant un certain nombre de génératrices distinctes et à chacune de ces génératrices correspond une courbe intégrale passant par le point de coordonnées x, y, z. Mais, si ce point vient sur la surface $P(x, y, z) = 0$, les deux cônes T et T' deviennent tangents suivant une génératrice commune G et, pour qu'il y eût une solution singulière passant par ce point, il faudrait évidemment que cette génératrice G fût située dans le plan tangent à la surface au point considéré, c'est-à-dire que l'on eût, pour tout point de cette surface,

$$\frac{\partial P}{\partial x} + \frac{\partial P}{\partial y} y' + \frac{\partial P}{\partial z} z' = 0, \tag{19}$$

y' et z' désignant les solutions communes des équations (15) et (16). Voici un moyen très-simple de s'assurer que cela n'a pas lieu en général. Remplaçons dans les équations (15) y' et z' par $y' - m$, $z' - n$, m et n désignant deux constantes quelconques ; nous ne changerons pas la surface $P(x, y, z)$ et la relation (19) deviendra

$$\frac{\partial P}{\partial x} + \frac{\partial P}{\partial y}(y' - m) + \frac{\partial P}{\partial z}(z' - n) = 0,$$

relation qui ne pourra être vérifiée pour des valeurs quelconques de m et de n.

Ainsi *le procédé qui devrait conduire aux solutions singulières, appliqué à un système quelconque de la forme* (15), *ne fournit aucune solution. Un système quel-*

conque d'équations différentielles simultanées n'admet pas d'une manière normale d'intégrales singulières.

Ce théorème est tout-à-fait analogue, comme on voit, au théorème connu sur les équations différentielles du premier ordre,* et le paradoxe auquel on est conduit s'explique de la même façon. En effet, pour établir les propriétés des surfaces focales d'une congruence de courbes, on suppose implicitement, il est aisé de s'en assurer, que les fonctions f et ϕ sont des fonctions continues de x, y, z, a, b dans le voisinage des valeurs des variables qui vérifient les relations

$$f=0,\ \phi=0,\frac{D(f,\phi)}{D(a,b)}=0.$$

Or, étant donné un systême quelconque d'équations différentielles de la forme (15), on sait bien qu'il admet une infinité d'intégrales dépendant de deux constantes arbitraires; mais rien ne prouve qu'on puisse mettre les équations des courbes intégrales sous la forme (13), les fonctions f et ϕ étant continues dans une étendue assez grande pour qu'on ait le droit d'appliquer la théorie des enveloppes. Nous pouvons même affirmer, d'après ce qui précède, qu'il n'en sera pas ainsi en général. Ainsi, tandis que les systêmes d'équations simultanées formées directement par l'élimination des constantes admettent d'une manière normale une infinité d'intégrales singulières, au contraire un systême d'équations différentielles pris arbitrairement ne pourra en avoir qu'à titre exceptionnel. Ceci nous montre qu'il est nécessaire, pour faire une théorie générale, de partir des équations différentielles elles-mêmes et non de leurs intégrales.

Il est bien clair d'ailleurs que les remarques précédentes n'enlèvent rien à l'intérêt des beaux théorèmes de Mr. Darboux sur les congruences de courbes, pas plus que nous ne devons rejeter la théorie des enveloppes parce que les solutions d'une équation différentielle du premier ordre n'admettent pas en général de courbe enveloppe.

7. Pour donner plus de précision aux raisonnements, je supposerai que les fonctions F et Φ sont des fonctions algébriques entières et irréductibles de x, y, z, y', z' et, comme il ne saurait être question de passer en revue tous les cas particuliers qui peuvent se présenter, je n'examinerai que les hypothèses les plus générales. Je négligerai d'ailleurs toutes les difficultés provenant de valeurs

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* *Bulletin des Sciences Mathématiques*, t. IV, 1ère Série, 1873.

infinies de y' et de z', difficultés qu'on peut toujours faire disparaître par un changement de coordonnées.

Pour un point quelconque de l'espace de coordonnées x_0, y_0, z_0, les deux cônes T, T' qui ont pour équations

$$F\left(x_0,\, y_0,\, z_0,\, \frac{Y-y_0}{X-x_0},\, \frac{Z-z_0}{X-x_0}\right)=0,$$
$$\Phi\left(x_0,\, y_0,\, z_0,\, \frac{Y-y_0}{X-x_0},\, \frac{Z-z_0}{X-x_0}\right)=0$$

se coupent suivant p génératrices distinctes. Soient

$$\frac{Y-y_0}{y'_0}=\frac{Z-z_0}{z'_0}=X-x_0$$

les équations d'une de ces génératrices ; le déterminant

$$\frac{D(F,\,\Phi)}{D(y',\,z')}$$

n'étant pas nul pour $x=x_0$, $y=y_0$, $z=z_0$, $y'=y'_0$, $z'=z'_0$, les équations (15) peuvent être résolues par rapport à y', z' et on en tire pour y', z' des développements en séries convergentes ordonnées suivant les puissances de $x-x_0$, $y-y_0$, $z-z_0$,

$$y'=y'_0+\alpha\,(x-x_0)+\beta\,(y-y_0)+\gamma\,(z-z_0)+\dots,$$
$$z'=z'_0+\alpha_1(x-x_0)+\beta_1(y-y_0)+\gamma_1(z-z_0)+\dots$$

On en déduit pour y et z les développements en série

$$y=y_0+y'_0(x-x_0)+\tfrac{1}{2}(\alpha+\beta y'_0+\gamma z'_0)(x-x_0)^2+\dots,$$
$$z=z_0+z'_0(x-x_0)+\tfrac{1}{2}(\alpha_1+\beta_1 y'_0+\gamma_1 z'_0)(x-x_0)^2+\dots$$

qui seront convergents dans un certain domaine du point x_0. Ainsi, *par un point quelconque de l'espace il passe en général p courbes intégrales, à tangentes distinctes, n'ayant en ce point aucune singularité.*

Il n'en est plus de même pour un point de coordonnées x_0, y_0, z_0 pris sur la surface $P(x,\,y,\,z)=0$ dont on obtient l'équation en éliminant y' et z' entre les équations (15) et (16). En effet, pour tout point de cette surface, les deux cônes T, T' deviennent tangents. Pour rester dans le cas le plus général, je supposerai que ces deux cônes sont simplement tangents le long d'une génératrice G et se coupent en outre suivant $p-2$ génératrices distinctes. A chacune de ces $p-2$ tangentes correspond une courbe intégrale n'ayant au point considéré aucune singularité. Il nous reste à rechercher les courbes intégrales tangentes à la droite G au point x_0, y_0, z_0.

Supposons qu'on ait pris pour origine des coordonnées le point x_0, y_0, z_0 lui-même et la droite G pour axe des x. Les équations (15) auront la forme suivante:

$$\left.\begin{aligned} A + By' + Cz' + Dy'^2 + \ldots\ldots &= 0, \\ A_1 + B_1y' + C_1z' + D_1y'^2 + \ldots\ldots &= 0, \end{aligned}\right\} \tag{20}$$

$A, A_1, B, B_1, \ldots\ldots$ désignant des polynômes entiers en x, y, z tels que

$$A = A_1 = BC_1 - CB_1 = 0 \text{ pour } x = y = z = 0.$$

Les deux cônes T, T', relatifs à l'origine des coordonnées, auront pour équations

$$(B)_0 \frac{Y}{X} + (C)_0 \frac{Z}{X} + \ldots\ldots = 0,$$
$$(B_1)_0 \frac{Y}{X} + (C_1)_0 \frac{Z}{X} + \ldots\ldots = 0,$$

où $(M)_0$ désigne, d'une manière générale, ce que devient M quand on y fait $x = y = z = 0$. Comme, par hypothèse, ces deux cônes sont simplement tangents, l'un au moins des coefficients $(B)_0$, $(C)_0$, $(B_1)_0$, $(C_1)_0$ sera différent de zéro. Supposons par exemple $(B)_0 \neq 0$; on pourra résoudre la première des équations (20) par rapport à y' et on en tirera pour y' une fonction holomorphe de x, y, z, z' s'annulant en même temps que ces variables

$$y' = -\frac{C}{B} z' + \ldots\ldots$$

et en portant cette valeur de y' dans la seconde des équations (20) on obtiendra une relation de la forme

$$A_1 + \left(C_1 - \frac{B_1C}{B}\right) z' + Kz'^2 + \ldots\ldots = 0.$$

Cette relation, considérée comme une équation en z', admet deux racines nulles et deux seulement pour $x = y = z = 0$. Ces deux valeurs de z' peuvent être considérées comme racines d'une équation du second degré dont les coefficients sont holomorphes en x, y, z dans le voisinage de l'origine. En résolvant cette équation et en portant la valeur de z' dans l'expression de y', on obtiendra finalement pour y' et z' des expressions de la forme

$$\left.\begin{aligned} y' &= A(x, y, z) + p(x, y, z)\sqrt{C(x, y, z)}, \\ z' &= B(x, y, z) + q(x, y, z)\sqrt{C(x, y, z)}, \end{aligned}\right\} \tag{21}$$

$A(x, y, z)$, $B(x, y, z)$, $C(x, y, z)$, $p(x, y, z)$, $q(x, y, z)$ étant des fonctions holomorphes de x, y, z dans le voisinage de l'origine, et les trois premières s'évanouissant à l'origine ; nous admettrons de plus, ce qu'on peut toujours faire, que le radical doit être pris avec la même détermination dans les deux formules. Ces deux systêmes de valeurs de y', z' deviennent égaux pour tout point dont les coordonnées vérifient la relation $C(x, y, z) = 0$; ceci nous montre que $C(x, y, z)$ doit être identique à $P(x, y, z)$ et, comme le plan tangent à cette surface $P(x, y, z) = 0$ ne doit pas contenir l'axe des x, on aura

$$C(x, y\ z) = ax + by + cz + dx^2 + \ldots.$$

le coefficient a étant différent de zéro. Soit

$$\begin{aligned} A(x, y, z) &= a_1x + b_1y + c_1z + d_1x^2 + \ldots., \\ B(x, y, z) &= a_2x + b_2y + c_2z + d_2x^2 + \ldots.; \end{aligned}$$

faisons le changement de variables

$$x = x'^2, \quad y = ux'^2, \quad z = vx'^2,$$

les équations (21) deviennent

$$x' \frac{du}{dx'} = -2u + 2x'^2(a_1 + b_1u + c_1v + \ldots) + 2x'p_1(x', u, v)\sqrt{a + bu + cv + \ldots},$$

$$x' \frac{dv}{dx'} = -2v + 2x'^2(a_2 + b_2u + c_2v + \ldots) + 2x'q_1(x', u, v)\sqrt{a + bu + cv + \ldots};$$

les seconds membres de ces deux équations sont des fonctions holomorphes pour $x' = u = v = 0$ puisque le coefficient a est essentiellement différent de zéro. D'après le théorème général, démontré au §1, ces équations admettent un systême unique d'intégrales s'évanouissant avec x', représentées par les séries convergentes

$$\begin{aligned} u &= \alpha x' + \beta x'^2 + \ldots., \\ v &= \alpha_1 x' + \beta_1 x'^2 + \ldots. \end{aligned}$$

En revenant aux variables primitives x, y, z, nous voyons que les équations de la courbe intégrale sont les suivantes :

$$\begin{aligned} x &= t^2, \\ y &= \alpha t^3 + \beta t^4 + \ldots., \\ z &= \alpha_1 t^3 + \beta_1 t^4 + \ldots.; \end{aligned}$$

cette courbe gauche possède un point de rebroussement à l'origine.

Ainsi, *la surface* $P(x, y, z) = 0$ *dont on obtient l'équation en éliminant* y' *et* z' *entre les trois équations*

$$F = 0, \quad \Phi = 0, \quad \frac{D(F, \Phi)}{D(y', z')} = 0$$

est en général le lieu des points de rebroussement des courbes intégrales.

En chaque point de cette surface passe une courbe intégrale ayant un rebroussement en ce point, et la tangente de rebroussement est la génératrice de contact des deux cônes T, T' *relatifs à ce point.*

8. Une intégrale des équations proposées sera dite singulière si pour chaque élément de cette intégrale on a à la fois

$$F = 0, \quad \Phi = 0, \quad \frac{D(F, \Phi)}{D(y', z')} = 0,$$

de sorte que les théorèmes généraux sur les équations différentielles ne s'appliquent plus à cette intégrale. Toute intégrale de cette espèce, s'il en existe, sera nécessairement située sur la surface $P(x, y, z) = 0$, obtenue en éliminant y' et z' entre les trois équations précédentes. Il faudra de plus que pour tout point de cette surface, au moins pour une certaine portion de cette surface, la génératrice de contact des deux cônes T, T', relatifs à ce point, soit située dans le plan tangent à la surface en ce point. Cette condition nécessaire est aussi suffisante; car, si elle est remplie, on aura pour définir les solutions singulières une équation différentielle du premier ordre. Les fonctions F et Φ étant données, on pourra donc toujours reconnaître par des calculs algébriques s'il existe des solutions singulières et former l'équation différentielle qui les caractérise.

Cette recherche pourra être facilitée dans certains cas par l'application de la règle suivante. Nous avons vu que pour tout point d'une solution singulière les équations (15), (16) et (18) devaient admettre une solution commune en y', z'. Réciproquement, supposons que pour tout point d'une certaine surface $Q(x, y, z) = 0$, (où $Q(x, y, z)$ sera nécessairement un diviseur de $P(x, y, z)$) les équations

$$\left.\begin{aligned}
&F(x, y, z, m, n) = 0, \ \Phi(x, y, z, m, n) = 0, \ \frac{D(F, \Phi)}{D(m, n)} = 0, \\
&\frac{D(F, \Phi)}{D(x, m)} + \frac{D(F, \Phi)}{D(y, m)} m + \frac{D(F, \Phi)}{D(z, m)} n = 0, \\
&\frac{D(F, \Phi)}{D(x, n)} + \frac{D(F, \Phi)}{D(y, n)} m + \frac{D(F, \Phi)}{D(z, n)} n = 0
\end{aligned}\right\} \qquad (22)$$

admettent une solution commune en m, n. Quand on se déplace sur cette surface, m et n sont des fonctions de deux variables indépendantes, de x et de y par exemple. En différentiant totalement les premières équations (22), on obtient

$$\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz + \frac{\partial F}{\partial m} dm + \frac{\partial F}{\partial n} dn = 0,$$

$$\frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy + \frac{\partial \Phi}{\partial z} dz + \frac{\partial \Phi}{\partial m} dm + \frac{\partial \Phi}{\partial n} dn = 0,$$

ou, en tenant compte de la relation $\frac{D(F, \Phi)}{D(m, n)} = 0$,

$$\frac{D(F, \Phi)}{D(x, m)} dx + \frac{D(F, \Phi)}{D(y, m)} dy + \frac{D(F, \Phi)}{D(z, m)} dz = 0,$$

$$\frac{D(F, \Phi)}{D(x, n)} dx + \frac{D(F, \Phi)}{D(y, n)} dy + \frac{D(F, \Phi)}{D(z, n)} dz = 0.$$

Admettons que l'un au moins des six déterminants fonctionnels qui figurent dans ces formules, par exemple $\frac{D(F, \Phi)}{D(x, m)}$, ne soit pas nul ; le plan tangent à la surface $Q(x, y, z) = 0$ aura pour équation

$$\frac{D(F, \Phi)}{D(x, m)}(X - x) + \frac{D(F, \Phi)}{D(y, m)}(Y - y) + \frac{D(F, \Phi)}{D(z, m)}(Z - z) = 0$$

et la quatrième des relations (22) exprime précisément que la droite

$$\frac{X - x}{1} = \frac{Y - y}{m} = \frac{Z - z}{n}$$

est située dans ce plan tangent ; et cette droite est la génératrice de contact des deux cônes T et T'. On peut donc énoncer la proposition suivante :

Lorsque pour tout point d'une surface $Q(x, y, z) = 0$ *les équations* (22) *admettent un système de solutions communes en* m, n, *sans que les six déterminants fonctionnels*

$$\frac{D(F, \Phi)}{D(x, m)}, \quad \frac{D(F, \Phi)}{D(y, m)}, \quad \frac{D(F, \Phi)}{D(z, m)}, \quad \frac{D(F, \Phi)}{D(x, n)}, \quad \frac{D(F, \Phi)}{D(y, n)}, \quad \frac{D(F, \Phi)}{D(z, n)}$$

soient tous nuls pour ces valeurs de m *et de* n, *les équations différentielles proposées admettent des solutions singulières.*

Les intégrales singulières, si elles existent, peuvent être considérées comme les enveloppes d'autres intégrales. En d'autres termes, par chaque point de la surface $Q(x, y, z) = 0$ il passe, outre une intégrale singulière, une intégrale non singulière tangente à la première. Imaginons que nous ayons pris pour origine des coordonnées un point de la surface $Q(x, y, z) = 0$ et pour plan des xy le plan tangent en ce point, et soit

$$z = \phi(x, y)$$

l'équation de cette surface. Si on change z en $z + \phi(x, y)$, il est clair qu'une transformation de cette nature n'altère pas les relations de contact entre deux courbes; on peut donc supposer, pour la commodité du raisonnement, que la surface lieu des solutions singulières se réduit au plan des xy. Les deux systêmes de valeurs de y', z' qui deviennent égaux pour $z = 0$ seront donnés par des formules de la forme suivante

$$\left.\begin{aligned} y' &= A(x, y, z) + p(x, y, z)\sqrt{z}, \\ z' &= B(x, y, z) + q(x, y, z)\sqrt{z}, \end{aligned}\right\} \tag{23}$$

$A(x, y, z)$, $B(x, y, z)$, $p(x, y, z)$, $q(x, y, z)$ désignant des séries convergentes ordonnées suivant les puissances croissantes de x, y, z. En outre, comme z' doit être nul pour tout point du plan des xy, $B(x, y, z)$ doit contenir z en facteur dans tous ses termes et on peut écrire

$$B(x, y, z) = zB_1(x, y, z).$$

Si dans les équations (23) on fait $z = 0$, elles se réduisent à l'équation unique

$$\frac{dy}{dx} = A(x, y, 0)$$

qui détermine les solutions singulières. Si on pose d'autre part $z = u^2$, elles deviennent

$$\begin{aligned} \frac{dy}{dx} &= A(x, y, u^2) + up(x, y, u^2), \\ 2\frac{du}{dx} &= uB_1(x, y, u^2) + q(x, y, u^2), \end{aligned}$$

et dans le voisinage de tout point $x = x_0$, $y = y_0$, $u = 0$, ces équations admettent un systême d'intégrales holomorphes pour lequel u n'est pas identiquement nul. *Par chaque point* (x_0, y_0) *du plan des* xy *passe donc une nouvelle courbe inté-*

grale tangente à la solution singulière passant par ce point. Ce résultat est bien conforme à la théorie des solutions singulières faite en partant des congruences de courbes et de leurs surfaces focales.

9. La proposition générale, qui fait le principal objet de ce travail, peut être établie sans avoir recours au théorème sur les équations différentielles qui a été démontré au début. Considérons d'abord une équation différentielle du premier ordre

$$F(x, y, y') = 0 \tag{24}$$

où F est un polynôme entier irréductible en x, y, y'; intégrer cette équation, cela revient à exprimer x, y, y' en fonction d'une seule variable indépendante de façon à vérifier l'équation (24) et la nouvelle relation

$$dy - y'dx = 0.$$

Or le système d'équations différentielles

$$\frac{dx}{\frac{\partial F}{\partial y'}} = \frac{dy}{y'\frac{\partial F}{\partial y'}} = \frac{dy'}{-\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y}y'} \tag{25}$$

admet l'intégrale première $F(x, y, y') = \text{const.}$, et si l'on choisit les valeurs initiales x_0, y_0, y'_0 de façon à ce qu'elles vérifient l'équation $F(x_0, y_0, y'_0) = 0$ on aura aussi pour les intégrales

$$F(x, y, y') = 0, \quad dy - y'dx = 0.$$

L'intégration de l'équation (24) revient donc à l'intégration du système (25), les valeurs initiales vérifiant la relation

$$F(x_0, y_0, y'_0) = 0.$$

Cela posé, supposons que pour $x = x_0$, $y = y_0$ l'équation (24) admette une racine multiple d'ordre n, $y' = y'_0$. Portons l'origine au point x_0, y_0 et prenons pour nouvel axe des x la droite de coefficient angulaire y'_0. On aura alors

$$F(x, y, y') = P(x, y) + y'P_1(x, y) + y'^2P_2(x, y) + \ldots + y'^nP_n(x, y) + \ldots$$

les n polynômes $P(x, y)$, $P_1(x, y), \ldots P_{n-1}(x, y)$ étant nuls pour $x = y = 0$, et $P_n(x, y)$ n'étant pas nul pour $x = y = 0$. Les équations (25) deviennent

$$\frac{dx}{P_1(x, y) + \ldots + (n-1)y'^{n-2}P_{n-2}(x, y) + \ldots} \\ = \frac{dy}{y'\{P_1(x, y) + \ldots\}} = \frac{dy'}{-\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y}y'}.$$

Plaçons-nous dans le cas général où il n'existe pas de solution singulière ; alors $\frac{\partial F}{\partial x} + y' \frac{\partial F}{\partial y}$ ne sera pas nul pour $x = y = y' = 0$; les équations précédentes pourront s'écrire

$$\frac{dx}{dy'} = Q_1(x, y) + y' Q_2(x, y) + \dots + y'^{n-2} Q_{n-1}(x, y) + y'^{n-1} Q_n(x, y) + \dots,$$
$$\frac{dy}{dy'} = y' Q_1(x, y) + y'^2 Q_2(x, y) + \dots + y'^{n-1} Q_{n-1}(x, y) + y'^n Q_n(x, y) + \dots,$$

les coefficients $Q_1, Q_2, \dots Q_{n-1}$ étant nuls pour $x = y = 0$. Ces équations admettent un système d'intégrales holomorphes s'évanouissant avec y', et il est aisé de voir que le développement de x commencera par un terme de degré n et celui de y par un terme de degré $n + 1$. La courbe intégrale sera donc représentée dans le voisinage de l'origine par les formules

$$x = a y'^n + \dots,$$
$$y = a_1 y'^{n+1} + \dots.$$

Si $n = 2$, ce qui est le cas général, la courbe présente un point de rebroussement de première espèce ; si $n > 2$, l'origine est un point singulier d'ordre plus élevé.

Etant donné un système d'équations simultanées

$$F(x, y, z, y', z') = 0, \quad \Phi(x, y, z, y', z') = 0,$$

on verra comme tout-à-l'heure que l'intégration de ce système revient à celui du système

$$\left.\begin{aligned} &\frac{dx}{\frac{D(F, \Phi)}{D(y', z')}} = \frac{dy}{y' \frac{D(F, \Phi)}{D(y', z')}} = \frac{dz}{z' \frac{D(F, \Phi)}{D(y', z')}} \\ &= \frac{dy'}{\frac{D(F, \Phi)}{D(z', x)} + y' \frac{D(F, \Phi)}{D(z', y)} + z' \frac{D(F, \Phi)}{D(z', z)}} = \frac{dz'}{\frac{D(F, \Phi)}{D(x, y')} + y' \frac{D(F, \Phi)}{D(y, y')} + z' \frac{D(F, \Phi)}{D(z, y')}} \end{aligned}\right\} , \quad (26)$$

les valeurs initiales $x_0, y_0, z_0, y'_0, z'_0$ vérifiant les relations

$$F(x_0, y_0, z_0, y'_0, z'_0) = 0, \quad \Phi(x_0, y_0, z_0, y'_0, z'_0) = 0.$$

Supposons qu'en un point de coordonnées x_0, y_0, z_0 les deux cônes T, T' soient tangents; si on choisit ce point pour origine et la génératrice de contact pour axe des x, on aura

$$\begin{aligned} F(x, y, z, y', z') &= A + By' + Cz' + Dy'^2 + 2Ey'z' + Fz'^2 + \ldots, \\ \Phi(x, y, z, y', z') &= a + by' + cz' + dy'^2 + 2ey'z' + fz'^2 + \ldots \end{aligned}$$

ou A, B, C, D, E, F, ... a, b, c, d, e, f, ... sont des polynômes entiers en x, y, z, A et a étant nuls pour $x = y = z = 0$. Désignons d'une manière générale par L_0 ce que devient une fonction quelconque L de x, y, z quand on y fait $x = y = z = 0$; les deux cônes T, T' auront pour équations

$$\begin{aligned} & B_0 \frac{Y}{X} + C_0 \frac{Z}{X} + D_0 \left(\frac{Y}{X}\right)^2 + 2E_0 \frac{YZ}{X^2} + F_0 \left(\frac{Z}{X}\right)^2 + \ldots, \\ & b_0 \frac{Y}{X} + c_0 \frac{Z}{X} + d_0 \left(\frac{Y}{X}\right)^2 + 2e_0 \frac{YZ}{X^2} + f_0 \left(\frac{Z}{X}\right)^2 + \ldots \end{aligned}$$

Pour que ces deux cônes soient tangents suivant l'axe des x il faut que l'on ait $B_0 c_0 - b_0 C_0 = 0$. Nous pouvons même supposer que l'on a pris le plan tangent commun pour plan des xy. On aura alors

$$B_0 = b_0 = 0,$$

et nous admettrons que C_0 et c_0 ne sont pas nuls; autrement l'axe des x serait une génératrice double pour l'un des cônes T, T'. Ecrivons les premiers termes des déterminants fonctionnels qui figurent dans les formules (26);

$$\begin{aligned} \frac{D(F, \Phi)}{D(y', z')} = {} & Bc - Cb + 2y'(Dc - Cd + Be - Eb) + 2z'(Ec - Ce + fB - bF) \\ & + 4(De - Ed)y'^2 + 4(Df - Fd)y'z' + 4(Ef - Fe)z'^2 + \ldots, \\ \frac{D(F, \Phi)}{D(x, z')} = {} & c\frac{\partial A}{\partial x} - C\frac{\partial a}{\partial x} + y'\left\{c\frac{\partial B}{\partial x} - C\frac{\partial b}{\partial x} + 2e\frac{\partial A}{\partial x} - 2E\frac{\partial a}{\partial x}\right\} \\ & + z'\left\{c\frac{\partial C}{\partial x} - C\frac{\partial c}{\partial x} + 2d\frac{\partial A}{\partial x} - 2D\frac{\partial a}{\partial x}\right\} + \ldots, \\ \frac{D(F, \Phi)}{D(x, y')} = {} & b\frac{\partial A}{\partial x} - B\frac{\partial a}{\partial x} + y'\left\{b\frac{\partial B}{\partial x} - B\frac{\partial b}{\partial x} + 2d\frac{\partial A}{\partial x} - 2D\frac{\partial a}{\partial x}\right\} \\ & + z'\left\{b\frac{\partial C}{\partial x} - B\frac{\partial c}{\partial x} + 2e\frac{\partial A}{\partial x} - 2E\frac{\partial a}{\partial x}\right\} + \ldots \end{aligned}$$

Pour $x = y = z = 0$, $\dfrac{D(F, \Phi)}{D(y', z')}$ se réduit à

$$2y'(D_0 c_0 - C_0 d_0) + 2z'(E_0 c_0 - C_0 e_0) + \ldots$$

Le coefficient de y' ne sera nul que si on a $D_0c_0 - C_0d_0 = 0$, et cette relation exprime, il est aisé de s'en assurer, que les deux cônes T et T' ont un contact du second ordre suivant l'axe des x. De même pour $x = y = z = y' = z' = 0$, les deux derniers dénominateurs des formules (26) se réduisent respectivement à

$$\left(c\frac{\partial A}{\partial x} - C\frac{\partial a}{\partial x}\right)_0$$

et à zéro. Prenons le cas général où il n'existe pas de solution singulière ; alors, comme nous l'avons vu, on aura

$$\left(c\frac{\partial A}{\partial x} - C\frac{\partial a}{\partial x}\right)_0 \neq 0$$

et les équations (26) pourront s'écrire

$$\left.\begin{aligned}
\frac{dx}{dy'} &= P_1 + P_2y' + P_3z' + \ldots\ldots,\\
\frac{dy}{dy'} &= P_1y' + P_2y'^2 + P_3y'z' + \ldots\ldots,\\
\frac{dz}{dy'} &= P_1z' + P_2y'z' + P_3z'^2 + \ldots\ldots,\\
\frac{dz'}{dy'} &= Q_1 + Q_2y' + Q_3z' + \ldots\ldots,
\end{aligned}\right\} \tag{27}$$

P_i et Q_i étant des fonctions holomorphes de x, y, z dans le domaine de l'origine. Dans le cas général où les deux cônes T et T' sont simplement tangents suivant l'axe des x, P_1 et Q_1 sont nuls à l'origine mais P_2 est différent de zéro. Les équations (27) admettent un système d'intégrales holomorphes s'évanouissant avec y' et on aura pour les premiers termes du développement

$$\begin{aligned}
x &= \alpha y'^2 + \ldots\ldots,\\
y &= \beta y'^3 + \ldots\ldots,\\
z &= \gamma y'^4 + \ldots\ldots,\\
z' &= \delta y'^2 + \ldots\ldots
\end{aligned}$$

Ces formules mettent en évidence le théorème général démontré plus haut (§7), mais nous voyons de plus que le plan tangent commun aux deux cônes T, T' rencontre la courbe intégrale en quatre points confondus avec l'origine.

Si on suppose maintenant que les deux cônes T et T' aient un contact du second ordre suivant l'axe des x, P_1, P_2 et Q_1 seront nuls pour $x = y = z = 0$, et on voit facilement que les développements des intégrales en séries auront la forme suivante

$$\begin{aligned} x &= \alpha y'^3 + \ldots\ldots, \\ y &= \beta y'^4 + \ldots\ldots, \\ z &= \gamma y'^5 + \ldots\ldots, \\ z' &= \delta y'^2 + \ldots\ldots \end{aligned}$$

La courbe intégrale présente à l'origine un point singulier d'espèce supérieure et le plan tangent commun aux deux cônes T, T' rencontre cette courbe en cinq points confondus à l'origine. Etant donné un système quelconque d'équations différentielles simultanées, les points singuliers précédents se présentent d'une manière normale. En effet, pour tout point de la surface $P(x, y, z) = 0$ les deux cônes T et T' ont en général un contact du premier ordre. Si on veut que ces deux cônes aient un contact du second ordre, il faudra joindre à l'équation $P(x, y, z) = 0$ une nouvelle équation de condition qui déterminera avec la première une courbe gauche pour tous les points de laquelle les deux cônes T et T' auront un contact du second ordre.

On examinerait de même le cas où les deux cônes ont un contact d'ordre quelconque, ou le cas où l'un des cônes admet une génératrice double appartenant à l'autre et en général tous les cas où $c\frac{\partial A}{\partial x} - C\frac{\partial a}{\partial x}$ n'est pas nul pour $x = y = z = 0$. Si on avait en même temps $\left(c\frac{\partial A}{\partial x} - C\frac{\partial a}{\partial x}\right) = 0$, tous les dénominateurs des équations (26) seraient nuls pour $x = y = z = y' = z' = 0$, et on ne pourrait plus appliquer la méthode précédente.

Il est facile de voir comment ces résultats se rattachent à la théorie des équations linéaires aux dérivées partielles. A chaque point de l'espace de coordonnées (x, y, z) les équations (15) font correspondre m directions issues de ce point; les surfaces telles que le plan tangent en chacun de leurs points contienne une des m droites correspondant à ce point seront définies par une équation aux dérivées partielles du premier ordre et du degré m en p, q, qui se décomposera en réalité en m équations linéaires. Les surfaces intégrales s'obtiennent, comme on sait, en associant suivant une loi arbitraire les courbes intégrales des équations (25). Si ces équations n'admettent pas de solutions singulières, comme c'est le cas général, l'équation aux dérivées partielles n'aura pas non plus de solution singulière. Mais, si les équations (15) admettent des solutions singulières, la surface engendrée par ces courbes donnera une solution singulière de l'équation aux dérivées partielles.

10. *Exemple* I.

$$y - xy' = 0, \quad x^2 z'^2 = x^2 + y^2 - 1.$$

Les deux valeurs de z' deviennent égales pour tous les points du cylindre

$$x^2 + y^2 - 1 = 0,$$

et la direction définie par la racine double est $z' = 0$, $y' = \frac{y}{x}$, c'est-à-dire la perpendiculaire abaissée du point (x, y) sur l'axe des z. Cette direction n'est pas située dans le plan tangent ou cylindre, qui est par conséquent un lieu de points de rebroussement des courbes intégrales. Il est facile de le vérifier car l'intégrale générale est

$$y = C_1 x, \quad z = \sqrt{x^2 + y^2 - 1} - \operatorname{arc\,tg} \sqrt{x^2 + y^2 - 1} + C_2.$$

Exemple II. (V. Serret : *Journal de Liouville*, 1^ère^ série, t. XVIII, p. 29).

$$y = xy' + y'^2 + z', \quad z = z'x + y'z'.$$

L'équation $\frac{D(F, \Phi)}{D(y', z')} = 0$ est ici

$$(x + y')(x + 2y') - z' = 0.$$

En éliminant y' et z' entre ces trois équations, on trouve l'équation d'une surface du sixième ordre

$$[2x^3 + 7xy - 9z]^2 + 2(x^2 + 3y)[12x(z - xy) - 2(x^2 - y^2)^3] = 0. \quad (28)$$

Il est aisé de voir que, pour tout point de cette surface, les équations (15), (16) et (18) sont compatibles ; nous nous trouvons donc dans le cas où il existe des solutions singulières. L'intégrale générale des équations proposées est en effet donnée par les équations

$$\left.\begin{aligned} y &= ax + a^2 + b, \\ z &= bx + ab, \end{aligned}\right\} \quad (29)$$

où a et b sont deux constantes arbitraires. Les courbes intégrales forment, comme on voit, un système de rayons rectilignes du second ordre et de la troisième classe ; ces droites sont des tangentes doubles de la surface (28), qui contiendra par conséquent les arêtes de rebroussement des développables de la congruence. Pour obtenir ces développables, cherchons à déterminer b en fonction de a de façon que les droites représentées par les équations (29) aient une enveloppe. On est conduit ainsi à l'équation différentielle

$$\left(\frac{db}{da}\right)^2 + a\frac{db}{da} - b = 0, \tag{30}$$

dont l'intégrale générale est

$$b = Ca + C^2,$$

C désignant une constante arbitraire. L'arête de rebroussement correspondante aura pour équations

$$\left.\begin{aligned} y &= -\frac{(x+C)^2}{4} + C^2, \\ z &= -\frac{C}{4}(x-C)^2, \end{aligned}\right\} \tag{31}$$

l'équation (30) admet en outre l'intégrale singulière,

$$b = -\frac{a^2}{4}$$

et l'arête de rebroussement correspondante est

$$\left.\begin{aligned} y &= -\frac{1}{3}x^3, \\ z &= -\frac{1}{27}x^3. \end{aligned}\right\} \tag{32}$$

Les équations proposées admettent, outre l'intégrale générale (29), une infinité d'intégrales singulières représentées par les équations (31) et une intégrale singulière isolée (32) qui est l'enveloppe des premières.

D'une manière générale, considérons le système d'équations simultanées

$$F(y - xy', z - xz', y', z') = 0, \quad \Phi(y - xy', z - xz', y', z') = 0,$$

qui peut être considéré comme une généralisation de l'équation de Clairaut. On vérifie immédiatement que les équations (15), (16) et (18) se réduisent à trois équations distinctes; le système doit donc admettre des intégrales singulières. En effet, l'intégrale générale est formée par les droites de la congruence

$$F(y-ax,\ z-bx,\ a,\ b)=0, \quad \Phi(y-ax,\ z-bx,\ a,\ b)=0,$$

où a et b sont des paramètres arbitraires; les arêtes de rebroussement des développables de la congruence seront les intégrales singulières.

III.

11. Les résultats qui précédent peuvent être appliqués aux équations différentielles du second ordre. Soit

$$F(x,\ y,\ y',\ y'')=0 \tag{33}$$

une équation du second ordre, où F est un polynôme entier irréductible en $x,\ y,\ y',\ y''$. Si on pose $y'=z$, l'équation (33) peut être remplacée par le système des deux équations

$$\left.\begin{aligned} F(x,\ y,\ z,\ z')=0,\\ y'-z=0.\end{aligned}\right\} \tag{34}$$

L'équation $P(x,\ y,\ z)=0$, obtenue en éliminant z' entre les deux équations

$$F(x,\ y,\ z,\ z')=0, \quad \frac{\partial F}{\partial z'}=0$$

représente en général, nous l'avons vu, le lieu des points de rebroussement des courbes intégrales du système (34). Si on y remplace z par y' on a une certaine équation différentielle du premier ordre

$$P(x,\ y,\ y')=0, \tag{35}$$

que l'on obtiendrait évidemment en éliminant y'' entre les deux équations

$$F(x,\ y,\ y',\ y'')=0, \quad \frac{\partial F}{\partial y''}=0,$$

et dont il est aisé, d'après ce qui précède, d'avoir la signification. En effet, soient $x,\ y,\ z$ les coordonnées d'un point M de la surface $P(x,\ y,\ z)=0$; par ce point

passe une courbe intégrale du système (34) ayant un rebroussement en ce point et la droite
$$\frac{X-x}{1}=\frac{Y-y}{y'}=\frac{Z-z}{z'}$$
pour tangente de rebroussement. Cette courbe se projette sur le plan des xy suivant une courbe intégrale de l'équation (33) passant par le point (x, y), ayant un rebroussement en ce point et la droite de coefficient angulaire $y'=z$ pour tangente de rebroussement. Par suite, en chaque point d'une courbe intégrale de l'équation (35) passe une courbe intégrale de l'équation (33) ayant un rebroussement en ce point et la tangente à la première courbe pour tangente de rebroussement.

Ce théorème peut s'établir directement comme il suit. Supposons, ce qu'on peut toujours faire grâce à un changement de coordonnées, que pour $x=y=y'=0$, l'équation (33) ait une racine double $y''=a$. Pour des valeurs de x, y, y' suffisamment voisines de zéro, les deux valeurs de y'' qui deviennent égales seront représentées par un développement de la forme
$$y''=a+(a_1x+b_1y+c_1y'+\dots)+\sqrt{a_2x+b_2y+c_2y'+\dots}\,; \tag{36}$$
en posant, comme plus haut, $y'=z$, cette équation peut être remplacée par le système
$$\left.\begin{aligned} z'&=a+(a_1x+b_1y+c_1z+\dots)+\sqrt{a_2x+b_2y+c_2z+\dots}\,,\\ y'&=z.\end{aligned}\right\}$$
Changeons encore z en $ax+Z$; le système devient
$$\left.\begin{aligned} Z'&=a_1x+b_1y+c_1Z+\dots+\sqrt{a_2x+b_2y+c_2Z+\dots}\,,\\ y'&=ax+Z.\end{aligned}\right\}$$
A ce système on peut appliquer le théorème du n° 1; car, si on pose
$$x=x'^2,\quad y=x'^2u,\quad Z=x'^2v,$$
il devient
$$\begin{aligned} \frac{x'}{2}\,\frac{du}{dx'}&=-u+ax'^2+x'^2v,\\ \frac{x'}{2}\,\frac{dv}{dx'}&=-v+x'^2\{a_1+b_1u+c_1v+\dots\}+x'\{a_2+\dots\}\,;\end{aligned}$$

ces équations admettent un système d'intégrales holomorphes s'évanouissant avec x'

$$v = \frac{2a_2}{3} x' + \dots\dots,$$

$$u = \frac{a}{2} x'^2 + \dots\dots$$

Par suite l'équation proposée admet une courbe intégrale tangente à l'axe des x à l'origine des coordonnées et représentée par les équations

$$x = x'^2, \quad y = \frac{a}{2} x'^4 + a x'^5 + \dots\dots$$

On voit que l'origine est un point de rebroussement de seconde espèce; ce qu'on pouvait prévoir *a priori* puisque la dérivée seconde doit conserver en ce point une valeur finie. En réunissant tous ces résultats, on peut énoncer la proposition ci-dessous:

Etant donnée une équation différentielle du second ordre $F(x, y, y', y'') = 0$, *en éliminant* y'' *entre cette équation et la relation* $\frac{\partial F}{\partial y''} = 0$, *on obtient une certaine équation différentielle du premier ordre* $P(x, y, y') = 0$, *dont les intégrales possèdent, en général, la propriété suivante. Par chaque point* M *d'une de ces courbes intégrales* C *il passe une courbe intégrale de l'équation* $F = 0$, *ayant un rebroussement de seconde espèce en* M *et la tangente en ce point à la courbe* C *pour tangente de rebroussement.*

On démontrerait aisément la proposition suivante, qui complète en quelque sorte la première:

En égalant à zéro le coefficient de la plus haute puissance de y'' *dans l'équation* $F(x, y, y', y'') = 0$, *on obtient une certaine équation différentielle du premier ordre* $Q(x, y, y') = 0$ *dont les courbes intégrales sont telles qu'en général par un point quelconque* M *de l'une d'elles* C *il passe une courbe intégrale de l'équation* $F = 0$, *ayant en* M *un rebroussement de première espèce et la tangente en ce point à la courbe* C *pour tangente de rebroussement.*

Si l'un des polynômes $P(x, y, y')$, $Q(x, y, y')$ ne contient pas y', on devra regarder, suivant les idées de Clebsch, l'intégrale correspondante comme se composant d'un point quelconque de la courbe $P(x, y) = 0$ ou $Q(x, y) = 0$ et de toutes les droites passant par ce point.

12. Nous avons supposé jusqu'ici que le système auxiliaire (34) n'admet pas de solutions singulières. S'il en est autrement, toutes ces solutions singulières seront situées sur la surface $P(x, y, z) = 0$ et leurs projections sur le plan des xy auront pour équation différentielle

$$P(x, y, y') = 0.$$

Par chaque point de la surface $P(x, y, z) = 0$ passent deux courbes intégrales du système (34) tangentes l'une à l'autre, dont les projections sur le plan des xy sont des courbes intégrales de l'équation (33) et, comme les valeurs de y' et de z' sont les mêmes pour les deux courbes dans l'espace, leurs projections sur le plan des xy auront un contact du second ordre. On voit donc que l'équation (33) admettra dans ce cas une infinité d'intégrales définies par une équation du premier ordre $P(x, y, y') = 0$ qui ont un contact du second ordre en chacun de leurs points avec une autre courbe intégrale.

Les solutions singulières de l'équation (33) peuvent aussi être définies directement, sans passer par l'intermédiaire du système simultané (34). Nous dirons qu'une intégrale de cette équation est singulière si pour tout point de cette intégrale la valeur correspondante de y'' est racine multiple de l'équation $F(x, y, y', y'') = 0$. Il résulte de cette définition que de pareilles intégrales, si elles existent, devront vérifier l'équation $P(x, y, y') = 0$ obtenue en éliminant y'' entre les équations $F = 0$ et $\frac{\partial F}{\partial y''} = 0$. On sera ramené à rechercher s'il existe des intégrales communes aux deux équations

$$F(x, y, y', y'') = 0, \quad P(x, y, y') = 0,$$

ce qu'on pourra toujours faire par des calculs algébriques. Si l'équation $F = 0$ est quelconque, ces deux équations n'auront pas d'intégrales communes et il n'existera pas de solutions singulières. Nous venons de voir quelle sera la propriété géométrique des courbes intégrales de l'équation $P = 0$. Mais, si les intégrales de l'équation $P = 0$ appartiennent à l'équation $F = 0$, on pourra démontrer directement, comme plus haut, que ces courbes ont en chacun de leurs points un contact du second ordre avec une autre courbe intégrale.

La recherche des solutions singulières peut être facilitée par l'application de la régle suivante. Il est clair d'abord que, s'il existe des solutions singulières, elles doivent satisfaire aux trois équations

$$F = 0, \quad \frac{\partial F}{\partial y''} = 0, \quad \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} y' + \frac{\partial F}{\partial y'} y'' = 0.$$

Inversement, si pour tous les points d'une courbe C ces trois équations admettent une solution commune en y'', cette courbe est une solution singulière. En effet soit m la racine commune aux trois équations

$$F(x, y, y', m) = 0, \quad \frac{\partial F}{\partial m} = 0, \quad \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} y' + \frac{\partial F}{\partial y'} m = 0;$$

quand on se déplace sur la courbe C, y, y', m sont des fonctions de x dont les dérivées vérifient la relation suivante

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} y' + \frac{\partial F}{\partial y'} y'' + \frac{\partial F}{\partial m} \cdot \frac{dm}{dx} = 0,$$

qui, comparée avec les premières, se réduit à

$$\frac{\partial F}{\partial y'} (y'' - m) = 0.$$

On aura donc $y'' = m$, à moins que l'on n'ait aussi $\frac{\partial F}{\partial y'} = 0$.

Remarque. Si l'équation $F = 0$ admet des solutions singulières définies par l'équation du premier ordre $P = 0$, cette équation $P = 0$ pourra avoir elle-même une solution singulière et cette solution singulière n'appartiendra pas en général à l'équation $F = 0$. Car cette solution n'a en général qu'un contact du premier ordre avec les autres intégrales de $P = 0$.

13. *Exemple* I. $$y''^2 - 2y'y'' + 1 = 0.$$

L'équation $P(x, y, y') = 0$ est ici $y'^2 - 1 = 0$; l'intégrale générale se compose de lignes droites

$$y = \pm x + C,$$

et il est facile de vérifier que ces lignes droites sont des lieux de points de rebroussement de seconde espèce pour les courbes intégrales de l'équation proposée ; l'intégrale générale, que l'on trouve aisément, est en effet représentée par les équations

$$\left.\begin{aligned} x &= \frac{p^2 - 1}{2} - \frac{1}{2} p\sqrt{p^2 - 1} + \frac{1}{2} L(p + \sqrt{p^2 - 1}) + C_1, \\ y &= \frac{p^3 - 1}{3} - \frac{(p^2 - 1)\sqrt{p^2 - 1}}{3} + C_2. \end{aligned}\right\}$$

Exemple II.

$$(1 + x^2) y''^2 - \left(2xy' + \frac{x^2}{2}\right) y'' + y'^2 + xy' - y = 0.* \qquad (37)$$

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* Lagrange, *Leçons sur le calcul des fonctions.* Serret, *Cours d'Analyse*, t. II, p. 395.

Les deux valeurs de y' deviennent égales si on a

$$y'^2 + \left(x + \frac{x^3}{2}\right) y' - (1 + x^2)\, y - \frac{x^4}{16} = 0. \tag{38}$$

On vérifie sans difficulté que les intégrales de l'équation (38) vérifient l'équation (37); il y a donc des solutions singulières. L'intégrale générale de l'équation (37) est en effet

$$y = ax^2 + bx + 4a^2 + b^2$$

et celle de l'équation (38)

$$\sqrt{16y + 4x^2 + x^4} - x\sqrt{1 + x^2} - \log\left(x + \sqrt{1 + x^2}\right) = H.$$

L'équation (38) admet en outre une intégrale singulière

$$y = -\frac{x^2}{4} - \frac{x^4}{16}$$

qui n'appartient pas à l'équation (37).

14. Il est aisé d'étendre ces considérations à des systêmes d'un nombre quelconque d'équations du premier ordre tels que

$$\left.\begin{array}{l} F_1(x, y_1, y_2, \ldots . y_n;\ y_1', y_2', \ldots . y_n') = 0, \\ F_2(x, y_1, y_2, \ldots . y_n;\ y_1', y_2', \ldots . y_n') = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ F_n(x, y_1, y_2, \ldots . y_n;\ y_1', y_2', \ldots . y_n') = 0. \end{array}\right\} \tag{39}$$

Appelons *point* dans l'espace à $(n + 1)$ dimensions tout systême de valeurs particulières $x_0, y_1^0, \ldots . y_n^0$ attribuées aux variables x, y_i, *courbe* l'ensemble des points dont les *coordonnées* sont fonctions d'une seule variable indépendante, *surface* l'ensemble des points qui vérifient une seule relation

$$\Phi(x, y_1, y_2, \ldots . y_n) = 0.$$

Tant que le déterminant fonctionnel

$$\frac{D(F_1, F_2, \ldots . F_n)}{D(y_1', y_2', \ldots . y_n')}$$

sera différent de zéro, on pourra résoudre les équations (39) par rapport à $y_1', y_2', \ldots . y_n'$ et appliquer le théorème fondamental de Cauchy. Mais il n'en sera plus de même si ce déterminant est nul. En éliminant $y'_1, \ldots . y'_n$ entre les équations (39) et l'équation

$$\frac{D(F_1, F_2, \ldots . F_n)}{D(y'_1, y'_2, \ldots . y'_n)} = 0, \tag{40}$$

on obtient une certaine surface

$$P(x, y_1, y_2, \ldots . y_n) = 0. \tag{41}$$

En général, par chaque point de cette surface passe une courbe intégrale présentant en ce point une singularité que l'on peut définir comme il suit; par une substitution linéaire convenable, $x, y_1, \ldots . y_n$ sont représentées par des développements de la forme

$$\begin{aligned} x &= t^2, \\ y_1 &= a_1 t^3 + \ldots . , \\ y_2 &= a_2 t^4 + \ldots . , \\ &\ldots\ldots\ldots\ldots \\ y_n &= a_n t^{n+2} + \ldots . \end{aligned}$$

Si, en chaque point de cette surface, la direction particulière définie par les équations (39) et (40) est située dans la *variété linéaire* à n dimensions tangente à cette surface, les conclusions sont tout-à-fait différentes. La surface $P = 0$ est un lieu d'intégrales singulières, que l'on peut définir par un système de $(n - 1)$ équations différentielles du premier ordre ou, ce qui revient au même, par une équation différentielle unique d'ordre $(n - 1)$.

Pour appliquer cette théorie à un exemple, reprenons un des problèmes traités par Serret;* proposons-nous de déterminer une courbe gauche, connaissant la courbe lieu des centres de courbure. Si on choisit x comme variable indépendante, on aura à rechercher les intégrales de deux équations différentielles du second ordre

$$\left.\begin{aligned} F(x, y, z, y', z', y'', z'') &= 0, \\ \Phi(x, y, z, y', z', y'', z'') &= 0; \end{aligned}\right\} \tag{42}$$

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* *Journal de Liouville*, 1ère Série, t. XVIII, p. 28.

ce système peut être remplacé par le suivant, qui ne contient que les dérivées du premier ordre

$$\left.\begin{aligned} &F(x, y, z, u, v, u', v') = 0, \\ &\Phi(x, y, z, u, v, u', v') = 0, \\ &y' - u = 0, \\ &z' - v = 0. \end{aligned}\right\} \tag{43}$$

L'intégrale générale doit contenir quatre constantes arbitraires; or on connaît immédiatement des courbes intégrales répondant à la question, les cercles qui ont leur centre en un point quelconque de la courbe donnée C. En dehors de ces solutions, évidentes *a priori*, il existe d'autres courbes répondant à la question que l'on pourra définir directement. Considérons une développable passant par la courbe C; cette développable sera définie si on se donne l'angle ϕ que fait en un point quelconque M de C le plan tangent à cette développable avec le plan osculateur à la courbe C en ce point. Soit D une développable obtenue ainsi, et G la génératrice passant en M. Toute courbe gauche répondant à la question peut évidemment être considérée comme l'arête de rebroussement de la surface enveloppe d'un plan P mené par M perpendiculairement à la génératrice G d'une certaine développable D; et il faudra de plus que le point de contact du plan P avec son enveloppe soit situé dans le plan tangent à la développable D suivant la génératrice G. En exprimant cette propriété, on est conduit, il est facile de s'en assurer, à une équation différentielle du troisième ordre dont les intégrales fournissent la véritable solution du problème proposé. Ces courbes correspondent à des solutions singulières du système (43); en effet, par chaque point de l'une d'elles passe une courbe intégrale ayant avec elle un contact du second ordre, le cercle osculateur lui-même, et pour ces deux intégrales x, y, z, u, v, y', z', u', v' auront les mêmes valeurs.

On peut encore s'en rendre compte autrement. Cherchons les courbes intégrales passant par un point donné M de l'espace et tangentes à une droite donnée MM' passant par ce point. Autrement dit, cherchons les intégrales du système (43) qui correspondent à des valeurs initiales données de x, y, z, u, v. Pour cela, au point M nous mènerons le plan P perpendiculaire à la droite MM', et nous prendrons les points d'intersection $0_1, 0_2, \ldots . 0_p$ de ce plan avec la courbe C; si ces p points sont distincts, ce qui est le cas général, les seules courbes répondant à la question seront les cercles décrits des points $0_1, 0_2, \ldots . 0_p$ comme centres et tangents à la droite MM'. Pour qu'il y ait une véritable courbe intégrale tangente en M à la droite MM', il faudra que le plan P soit tangent à la courbe C; par suite deux des systêmes de valeurs de u', v', fournies par les équations (43) seront venu se confondre: ce qui est la propriété caractéristique des solutions singulières.

15. Prenons encore une équation unique d'ordre n

$$F(x, y, y', y'', \ldots . y^{(n)}) = 0, \tag{44}$$

ou F désigne une fonction entière de x, y, y', ... $y^{(n)}$; si, pour

$$x = x_0, \ y = y_0, \ldots . y^{n-1} = y_0^{n-1};$$

l'équation (44) admet une racine multiple $y^n = y_0^n$, on ne peut plus appliquer les théorèmes généraux. Pour trouver la singularité correspondante de l'intégrale, imaginons, ce qu'on peut toujours faire, que cette circonstance se présente pour les valeurs initiales $x_0 = y_0 = y_0' = 0$; et supposons d'abord que l'on ait aussi $y_0'' = y_0''' = \ldots . = y_0^n = 0$, la valeur de $y^{(n)}$ qui se réduit à zéro étant racine double seulement. Les deux valeurs de $y^{(n)}$ qui deviennent nulles seront représentées par un développement de la forme

$$y^{(n)} = a_1x + a_2y + a_3y' + \ldots + a_{n+1}y^{(n-1)} + \ldots + \sqrt{b_1x + b_2y + b_3y' + \ldots} \tag{45}$$

Posons
$$y' = u_1, \ y'' = u_2, \ldots . y^{(n-1)} = u_{n-1};$$

on peut remplacer l'équation (45) par le systême

$$\left.\begin{aligned} y' &= u_1, \\ u_1' &= u_2, \\ &\ldots\ldots \\ u_{n-2}' &= u_{n-1}, \\ u_{n-1}' &= a_1x + a_2y + a_3u_1 + \ldots . + \sqrt{b_1x + b_2y + b_3u_1 + \ldots .} \end{aligned}\right\} \tag{46}$$

Faisons encore le changement de variables

$$x = x'^2, \ y = x'^2 Y, \ u_1 = x'^2 U_1, \ldots . u_{n-1} = x'^2 U_{n-1};$$

le système (46) devient

$$\left.\begin{aligned}
&\frac{x'}{2}\,\frac{dY}{dx'}+Y=x'^{n}U_1,\\
&\dots\dots\dots\dots\dots\\
&\frac{x'}{2}\,\frac{dU_{n-2}}{dx'}+U_{n-2}=x'^{n}U_{n-1},\\
&\frac{x'}{2}\,\frac{dU_{n-1}}{dx'}+U_{n-1}=x'^{n}\{a+a_1Y+\dots\dots\}+x'\,\{b^{\frac{1}{2}}+\dots\dots\}
\end{aligned}\right\}\qquad(47)$$

D'après le théorème général du n° 1, ce système admet un système d'intégrales holomorphes s'évanouissant avec x', et on vérifie aisément que le développement de Y commencera par un terme en $(x')^{2n+1}$. On aura donc pour y un développement suivant les puissances de $x^{\frac{1}{2}}$ de la forme suivante

$$y=\alpha x^{n+\frac{1}{2}}+\beta x^{n+1}+\gamma x^{n+\frac{3}{2}}+\dots\dots$$

Supposons maintenant que pour $x=y=y'=0$, $y''=y_0''\dots\dots y^{(n-1)}=y_0^{(n-1)}$, l'équation (44) admette une racine double $y^{(n)}=y_0^{(n)}$. On ramènera ce cas au précédent en posant

$$y=\frac{y_0''}{2}x^2+\frac{y_0'''}{1.2.3}x^3+\dots\dots+\frac{y_0^{(n)}}{1.2\dots\dots n}x^n+z;$$

on en conclut que le développement de y sera de la forme

$$y=\frac{y_0''}{2}x^2+\frac{y_0'''}{1.2.3}x^3+\dots\dots+\frac{y_0^{(n)}}{1.2\dots\dots n}x^n+\alpha x^{n+\frac{1}{2}}+\beta x^{n+1}+\dots\dots\qquad(48)$$

La courbe intégrale présente à l'origine un point singulier d'espèce particulière, analogue à un point de rebroussement de seconde espèce.

Pour que l'équation (44) ait une racine double en $y^{(n)}$, il faut que l'on ait en même temps

$$\frac{\partial F}{\partial y^{(n)}}=0;$$

l'élimination de $y^{(n)}$ entre ces deux équations conduit à une relation entre y et ses dérivées jusqu'à l'ordre $n-1$.

$$\Phi(x,\,y,\,y',\dots\dots y^{(n-1)})=0.\qquad(49)$$

D'après ce que nous venons de voir, les courbes intégrales de cette équation (49) possèdent la propriété suivante. En chaque point M de l'une d'elles C, *il passe une courbe intégrale de l'équation proposée* (44) *ayant avec la courbe* C *un contact d'ordre* $n - 1$ *et présentant en ce point un point singulier de l'espèce caractérisée par le développement* (48).

Si les intégrales de l'équation (49) vérifient aussi l'équation (44), ce qui ne peut avoir lieu que dans des cas particuliers, ces intégrales seront des intégrales *singulières*. On démontrerait comme précédemment qu'en chaque point de l'une d'elles passe une seconde courbe intégrale ayant un contact d'ordre n avec l'intégrale singulière.

PARIS, Janvier 1889.

Electromagnetic Waves and Oscillations at the Surface of Conductors.

BY HENRY A. ROWLAND.

GENERAL EQUATIONS.

In the following paper I have worked out a few cases of electromagnetic waves and the oscillations of electricity on a conducting body, such as may be useful in the further understanding of alternating currents and the subject of electricity generally.

Of course all calculations must be based on Maxwell's equations. In these equations occur two quantities, J and ψ, which caused remarks by Sir Wm. Thomson and others at the British Association meeting in Bath. Maxwell has already given the reasons for rejecting ψ, and has shown that neither J nor ψ enter into the theory of waves. In order, however, that there shall be no propagation of free electricity in a non-conductor, the components of the electric force must satisfy the equation of continuity, and this leads to components of the vector potential satisfying the same equation, and $J=0$ therefore. I have satisfied myself that there is absolutely no loss of generality from these changes. Hence I write the equations as follows: F, G, H being the components of the vector potential, u, v, w of the electric current, P, Q, R of the electric force, $\bar{a}$, $\bar{b}$, $\bar{c}$ of the magnetic induction, κ the specific inductive capacity, μ the magnetic permeability, C the conductivity, t the time and σ the surface density.

$$\bar{a}=\frac{\partial H}{\partial y}-\frac{\partial G}{\partial z}; \qquad P=-\frac{dF}{dt}$$

$$\bar{b}=\frac{\partial F}{\partial z}-\frac{\partial H}{\partial x}; \qquad Q=-\frac{dG}{dt}$$

$$\bar{c}=\frac{\partial G}{\partial x}-\frac{\partial F}{\partial y}; \qquad R=-\frac{dH}{dt}$$

$$u = -\frac{1}{4\pi\mu}\Delta^2 F = -\left(C + \frac{\kappa}{4\pi}\frac{d}{dt}\right)\frac{dF}{dt}$$

$$v = -\frac{1}{4\pi\mu}\Delta^2 G = -\left(C + \frac{\kappa}{4\pi}\frac{d}{dt}\right)\frac{dG}{dt}$$

$$w = -\frac{1}{4\pi\mu}\Delta^2 H = -\left(C + \frac{\kappa}{4\pi}\frac{d}{dt}\right)\frac{dH}{dt}$$

$$\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} = 0.$$

Inside a good conductor, such as a metal, Maxwell has shown that κ can be neglected in comparison with the conductivity, and the equations take the form of the diffusion equations for heat.

The conditions at a surface of separation of two media are as follows, though all the equations are not independent, l, m and n being the direction cosines of the normal to the surface:

$$\begin{aligned}
&(\bar{a} - \bar{a}_1)\,l + (\bar{b} - \bar{b}_1)\,m + (\bar{c} - \bar{c}_1)\,n = 0,\\
&(u - u_1)\,l + (v - v_1)\,m + (w - w_1)\,n = 0,\\
&(\kappa P - \kappa_1 P_1)\,l + (\kappa Q - \kappa_1 Q_1)\,m + (\kappa R - \kappa_1 R_1)\,n = 4\pi\sigma,
\end{aligned}$$

$$\begin{aligned}
&\left(\frac{\bar{b}}{\mu} - \frac{\bar{b}_1}{\mu_1}\right)n - \left(\frac{\bar{c}}{\mu} - \frac{\bar{c}_1}{\mu_1}\right)m = 0,\\
&\left(\frac{\bar{c}}{\mu} - \frac{\bar{c}_1}{\mu_1}\right)l - \left(\frac{\bar{a}}{\mu} - \frac{\bar{a}_1}{\mu_1}\right)n = 0,\\
&\left(\frac{\bar{a}}{\mu} - \frac{\bar{a}_1}{\mu_1}\right)m - \left(\frac{\bar{b}}{\mu} - \frac{\bar{b}_1}{\mu_1}\right)l = 0,
\end{aligned}$$

$$\begin{aligned}
&(Q - Q_1)\,n - (R - R_1)\,m = 0,\\
&(R - R_1)\,l - (P - P_1)\,n = 0,\\
&(P - P_1)\,m - (Q - Q_1)\,l = 0.
\end{aligned}$$

The electrostatic energy of any volume is

$$\frac{\kappa}{8\pi}\int\!\!\int\!\!\int (P^2 + Q^2 + R^2)\,dx\,dy\,dz,$$

and the electromagnetic energy is

$$\frac{1}{8\pi\mu}\int\!\!\int\!\!\int (\bar{a}^2 + \bar{b}^2 + \bar{c}^2)\,dx\,dy\,dz.$$

The equations of Maxwell, applied to periodic disturbance, indicate that all the vector quantities have only two directions in space at any point. The vector potential, the electric current, the electric force and the electric induction being in one direction, and the magnetic force and magnetic induction being in another direction, the cosine of the angle between these directions being

$$\frac{F\bar{a} + G\bar{b} + H\bar{c}}{\sqrt{(\bar{a}^2 + \bar{b}^2 + \bar{c}^2)(F^2 + G^2 + H^2)}}.$$

A system of surfaces can always be drawn containing these two vectors, as in hydrodynamics in the case of steady flow for the vortex lines and lines of flow, and these surfaces constitute the wave surface when the motion is periodic. In any case we may call them by this name. In the case of periodic motion, the flow of energy is perpendicular to this surface for plane waves at least, and Prof. Poynting has carried out the idea for all cases and supposes the energy always to flow perpendicular to this surface.

At the surface of a perfect conductor the conditions are much simplified. The disturbance penetrates only an infinitely small distance into the surface, and consists of a current sheet whose components per unit length along arc of cross section can be designated by U, V, W. The surface conditions are then transformed into the following:

$$U = \frac{1}{4\pi\mu}\{m\bar{c} - n\bar{b}\},$$

$$V = \frac{1}{4\pi\mu}\{n\bar{a} - l\bar{c}\},$$

$$W = \frac{1}{4\pi\mu}\{l\bar{b} - m\bar{a}\},$$

$$\therefore \left\{\begin{aligned} &\bar{a}U + \bar{b}V + \bar{c}W = 0, \\ &lU + mV + nW = 0, \end{aligned}\right.$$

$$lu + mv + nw = -\left\{\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}\right\} = -\frac{d\sigma}{dt},$$

$$l\bar{a} + m\bar{b} + n\bar{c} = 0,$$

$$lP + mQ + nR = \frac{4\pi\sigma}{\kappa},$$

$$mR - nQ = 0,$$

$$nP - lR = 0,$$

$$lQ - mP = 0.$$

From these equations we see that the electrical vectors, such as the vector potential, electric force and electric current, in the medium must be perpendicular to the perfectly conducting surface, and that the magnetic force and induction must be parallel to the surface. The wave surface must then be perpendicular to the conducting surface, and the currents in the surface are in the direction of the normal to the wave surface. Thus the system is orthogonal at the surface.

Selecting a complete orthogonal system, one surface being the wave surface, we can replace the surface containing the magnetic induction $\bar{a}$, $\bar{b}$, $\bar{c}$ by a perfectly conducting surface.

The solution of cases of progressive waves, as well as stationary waves and electric oscillations on conductors, are all dependent on these general equations. One general fact with regard to electric oscillations is to be noted, and that is, that, as in the case of sound, the period of oscillation of an electric system is proportional to the linear dimensions of the system.

TWO DIMENSIONS.

Suppose the waves to move in the direction of the axis of X without any change of form, but with or without damping, which would influence the amplitude of the disturbance. Then the disturbance can be expressed by the equations

$$\begin{aligned} F &= -\frac{1}{a} L\varepsilon^{ax+ct}, & \bar{a} &= T\varepsilon^{ax+ct}, \\ G &= M\varepsilon^{ax+ct}, & \bar{b} &= \frac{1}{a}\left(\frac{\partial T}{\partial y} - AN\right)\varepsilon^{ax+ct}, \\ H &= N\varepsilon^{ax+ct}, & \bar{c} &= \frac{1}{a}\left(\frac{\partial T}{\partial z} + AM\right)\varepsilon^{ax+ct}, \end{aligned}$$

where L, M, N are functions of y and z and

$$A = c\mu\,(4\pi C + c\kappa); \quad L = \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z}; \quad T = \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z};$$
$$\Delta^2 M + (a^2 - A)\,M = 0; \quad \Delta^2 N + (a^2 - A)\,N = 0.$$

In these equations in general $a = \alpha + i\beta$ and $c = h - ib$, where α is the distance damping factor, h is the time damping factor, the velocity of the wave is $\frac{b}{\beta}$ and $b = \pi\nu$, where ν is the number of alternations per second.

A special case is given by $F=0$ and $a^2-A=0$, which is applicable to waves on perfectly conducting cylinders.

These conditions give

$$\frac{\partial M}{\partial z}-\frac{\partial N}{\partial y}=0,$$

$$\frac{\partial M}{\partial y}+\frac{\partial N}{\partial z}=0.$$

Hence M and N are conjugate functions of y and z. If ϕ and ψ are conjugate functions of y and z, we can therefore write

$$M=\frac{\partial \phi}{\partial y}=\quad\frac{\partial \psi}{\partial z},$$

$$N=\frac{\partial \phi}{\partial z}=-\frac{\partial \psi}{\partial y},$$

$$\begin{aligned} \therefore\ F&=0, & \bar{a}&=0,\\ G&=\frac{\partial \phi}{\partial y}\,\epsilon^{ax+ct}, & \bar{b}&=a\frac{\partial \psi}{\partial y}\,\epsilon^{ax+ct},\\ H&=\frac{\partial \phi}{\partial z}\,\epsilon^{ax+ct}, & \bar{c}&=a\frac{\partial \psi}{\partial z}\,\epsilon^{ax+ct}. \end{aligned}$$

Hence the electric quantities lie on the curve $\psi=$ const. and the magnetic ones on $\phi=$ const. Therefore the conditions at the surface of a perfect conductor are satisfied for the cylindrical surfaces $\phi=$ const.

In general, we take two cylinders, ϕ' and ϕ'', so as to account for the direct and return currents and for the + and − charges.

The surface density on $\phi=$ const. is found to be

$$\sigma=\frac{-c\kappa}{4\pi}\,h\epsilon^{ax+ct},$$

where

$$h^2=\left(\frac{\partial \phi}{\partial y}\right)^2+\left(\frac{\partial \phi}{\partial z}\right)^2=\left(\frac{\partial \psi}{\partial y}\right)^2+\left(\frac{\partial \psi}{\partial z}\right)^2$$

and the electric current

$$U_1=a\,\frac{h}{4\pi\mu}\,\epsilon^{ax+ct}.$$

Hence we have the remarkable theorems:

Theorem I. *The electric density on the cylinders is proportional to the density which they would have if they were charged with electricity at rest; and the electric currents are distributed in the same proportion, although not the same phase.*

This leads to

Theorem II. *In the case of perfect conductors we have* $LC = \frac{1}{v^2}$, *where* L *is the self-induction and* C *the capacity per unit of length, and* v *the ratio of the units or the velocity of light.*

This is best proved by considering the energy of the system per unit of length as derived by the formulae above and by the ordinary ones in terms of the capacity and self-induction. Thus we have

$$\frac{1}{2}\frac{\left(\int \sigma ds\right)^2}{C} = \frac{\kappa c^2}{8\pi}\int\int\int h^2 (\varepsilon^{ax+ct})^2\, dxdydz,$$

$$\frac{1}{2}L\left(\int U_1 ds\right)^2 = \frac{a^2}{8\pi\mu}\int\int\int h^2 (\varepsilon^{ax+ct})^2\, dxdydz,$$

where ds is an element of the curve ϕ constant, and the line integrals are taken around the curve ϕ' or ϕ'', and the volume integrals between ϕ' and ϕ''. Whence, substituting the values of σ and U_1 and dividing one equation by the other, the theorem is proved.

In the case of imperfect conductors and long waves, the currents sink below the surface and the self-induction becomes greater than $\frac{1}{Cv^2}$.

It is to be noted that rapid alternation of the current to some extent takes the place of improved conductivity in causing a superficial distribution of the current. Indeed the equations only contain the conductivity multiplied by the number of alternations per second, although the number of alternations per second enters separately. Within certain limits, therefore, the following is correct:

Starting with very good conductors and very long waves, the electric current will be uniformly distributed throughout the section of the conductors. As the waves become shorter and the number of vibrations per second greater, the distribution of currents changes and finally they are entirely on the surface of the conductors and distributed very nearly like the density of electricity, provided the conductors were charged with electricity at rest. The waves must, however, be long as compared with the sectional area of the cylinders.*

Plane Waves at the Surface of Separation of Two Media.

At the surface of a liquid or of an elastic solid we may have waves propagated according to known laws. So, at the surface separating two media, there may be waves whose theory will now be given. In this investigation a, b, c, e and A, B will be complex constants. Let

$$\begin{aligned} F &= Ae\varepsilon^{ax+ey+ct}, \\ G &= -Aa\varepsilon^{ax+ey+ct}, \\ H &= B\,\varepsilon^{ax+ey+ct}, \end{aligned}$$

which satisfy the equation of continuity. Maxwell's equation gives also

$$-\mu c(4\pi C - \varkappa c) + a^2 + e^2 = 0.$$

Let the plane $y = 0$ separate the two media of conductivities C and C_1 and specific inductive capacities and magnetic permeabilities $\varkappa\mu$ and $\varkappa_1\mu_1$. Maxwell's equation and the surface conditions then give

$$\begin{aligned} a &= a_1, & c &= c_1, \\ a^2 + e^2 &= \mu c(4\pi C + \varkappa c), & a^2 + e_1^2 &= \mu_1 c(4\pi C_1 + \varkappa_1 c), \\ \frac{A}{\mu}(a^2 + e^2) &= \frac{A_1}{\mu_1}(e_1^2 + a^2), & B &= B_1, \\ Ae &= A_1 e_1, & \frac{e}{\mu} &= \frac{e_1}{\mu_1}. \end{aligned}$$

There are two principal cases—

1*st.—Magnetic Waves.*

The condition is $A = A_1 = 0$,

$$F = G = 0, \qquad F_1 = G_1 = 0.$$

$$H = B\varepsilon^{ax+ey+ct}, \qquad H_1 = B\varepsilon^{ax+\frac{\mu_1}{\mu}ey+ct}.$$

When there is no time damping at the source of the wave, we can pass to a real solution as follows:

$$a = \alpha + i\beta, \quad e = \gamma + i\delta, \quad c = -ib = -i\pi\nu.$$

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* The distribution of currents is superficial for moderate conductivity and short waves. Hence, inside the conductors, the electric and magnetic forces must be zero, and this leads to the same superficial distribution of electric currents and surface density which, when the section of the cylinder is small compared with the wave length, becomes that of electricity at rest.

The conditions from Maxwell's equations give us the values of these in terms of b.

$$\alpha^2 = b\,\frac{\mu\mu_1}{2(\mu^2-\mu_1^2)}\left\{\sqrt{16\pi^2(C_1^{\mu}-C_{\mu_1})^2+b^2(\varkappa_1^{\mu}-\varkappa_{\mu_1})^2}+b(\varkappa_1^{\mu}-\varkappa_{\mu_1})\right\},$$

$$\beta^2 = b\,\frac{\mu\mu_1}{2(\mu^2-\mu_1^2)}\left\{\sqrt{16\pi^2(C_1^{\mu}-C_{\mu_1})^2+b^2(\varkappa_1^{\mu}-\varkappa_{\mu_1})^2}-b(\varkappa_1^{\mu}-\varkappa_{\mu_1})\right\},$$

$$\gamma^2 = b\,\frac{\mu^2}{2(\mu^2-\mu_1^2)}\left\{\sqrt{16\pi^2(C_1\mu_1-C\mu)^2+b^2(\varkappa_1\mu_1-\varkappa\mu)^2}+b(\varkappa_1\mu_1+\varkappa\mu)\right\},$$

$$\delta^2 = b\,\frac{\mu^2}{2(\mu^2-\mu_1^2)}\left\{\sqrt{16\pi^2(C_1\mu_1-C\mu)^2+b^2(\varkappa_1\mu_1-\varkappa\mu)^2}-b(\varkappa_1\mu_1-\varkappa\mu)\right\},$$

where α and β must have opposite signs. For the second medium

$$\alpha_1=\alpha;\quad \beta_1=\beta;\ \gamma_1=\frac{\mu_1}{\mu}\gamma;\quad \delta_1=\frac{\mu_1}{\mu}\delta.$$

When $\mu_1=\mu$, these all become infinite and no wave is propagated. Hence they depend on the difference of magnetic properties. The magnetic disturbance is in the plane xy and the electrical disturbance in the direction of the axis of z. The wave advances in direction of x with a velocity $\frac{b}{\beta}$ and a damping factor α. The waves in both media die out quickly as one passes away from the surface according to the damping factors γ and γ_1. So that the wave is confined to the surface in the same sense that a water wave is confined to the surface of water.

The case of greatest consequence is that of iron and air for which

$$C=0,\quad \mu=1,\quad \varkappa=\frac{1}{9\times10^{20}};\quad C_1=\frac{1}{10000},\quad \mu_1=1000,\quad \varkappa_1=0,$$

all on the c. g. s. system. These give very nearly

$$-\alpha=\beta=-\gamma=\delta=\pi\sqrt{\frac{2C_1\nu}{\mu}}\ \text{ and }\ \gamma_1=-\delta_1=\pi\sqrt{2C_1\mu_1\nu}.$$

If λ is the wave length of a complete wave, ν the number of reversals per second and V the velocity of the waves, we have

$$b=\frac{2\pi V}{\lambda}=\beta V=\pi V=\beta\frac{\nu\lambda}{2}.$$

Hence the disturbance is given by the following quantities:

Vector Potential.

In Air.

$$F=G=0,$$
$$H=B\varepsilon^{-\beta(x+y)}\cos\{\beta(x+y)-\pi\nu t\}.$$

In the Magnetic Metal.

$$F_1 = G_1 = 0,$$
$$H_1 = B\epsilon^{-\beta(x+\mu_1 y)} \cos \{\beta (x + \mu_1 y - \pi\nu t\}.$$

Magnetic Induction.

In Air.

$$\bar{a} = - B\epsilon^{-\beta(x+y)}\beta \{\sin [\beta (x + y) - \pi\nu t] + \cos [\beta (x + y) - \pi\nu t]\},$$
$$\bar{b} = + B\epsilon^{-\beta(x+y)}\beta \{\sin [\beta (x + y) - \pi\nu t] + \cos [\beta (x + y) - \pi\nu t]\},$$
$$\bar{c} = 0.$$

In the Magnetic Metal.

$$\bar{a}_1 = - B\epsilon^{-\beta(x+\mu_1 y)}\beta\mu_1 \{\sin [\beta (x + \mu_1 y) - \pi\nu t] + \cos [\beta (x + \mu_1 y) - \pi\nu t]\},$$
$$\bar{b}_1 = + B\epsilon^{-\beta(x+\mu_1 y)}\beta \{\sin [\beta (x + \mu_1 y) - \pi\nu t] + \cos [\beta (x + \mu_1 y) - \pi\nu t]\},$$
$$\bar{c}_1 = 0.$$

The waves which proceed outward from the surface into the two media are therefore of the same type as the heat waves sent downward into the earth from the periodic heating of the sun, and may be called diffusion waves. The component along the surface is of the same type. In this type the real and imaginary parts of the coefficients of x or y are equal, and therefore the amplitude of the wave is decreased to $\frac{1}{535}$th of its value in going a complete wave length or $\frac{1}{23}$rd in going half a wave length. As an example, take iron with $\nu = 200.$ per second. We then have

Velocity along surface, $\frac{b}{\beta} = \sqrt{\frac{\nu\mu_1}{2C_1}} = 31600.$ cm. per sec.

Wave length along surface, $\frac{2\pi}{\beta} = \sqrt{\frac{2\mu_1}{\nu C_1}} = 316.$ cm.

Perpendicular velocity in air, $\frac{b}{\delta} = \frac{b}{\beta} = \sqrt{\frac{\nu\mu_1}{2C_1}} = 31600.$ cm. per sec.

" wave length in air, $\frac{2\pi}{\delta} = \frac{2\pi}{\beta} = \sqrt{\frac{2\mu_1}{\nu C_1}} = 316.$ cm.

" velocity in iron, $\frac{b}{\delta_1} = \frac{b}{\mu_1\beta} = \sqrt{\frac{\nu}{2C_1\mu_1}} = 32.$ cm. per sec.

" wave length in iron, $\frac{2\pi}{\delta_1} = \frac{2\pi}{\mu_1\beta} = \sqrt{\frac{2}{\nu C_1\mu_1}} = .32$ cm.

One is impressed with the small values of all these quantities, especially the velocity of penetration into the iron, which is only 32. cm. per second. At a depth of one-fourth λ the disturbance is only one-fifth that at the surface, and in this case it is only .08 cm. or $\frac{1}{32}$ inch! Even at the depth of $\frac{1}{60}$ inch the disturbance is only about half that at the surface. The disturbance along the surface is also only half as great when the distance from the disturbing cause is 40 cm. or about 8 inches.

This theory explains an experiment of Prof. Trowbridge, in which a bar was placed in a solenoid through which an alternating current was passed. Only the outside was found to be magnetized and to attract iron filings.

From this theory we see that any iron requiring very rapid changes of magnetization must be divided up into very fine wires or laminae, in the well-known manner, whose thickness or diameter can be reduced with advantage even below $\frac{1}{100}$ inch.

The lines of magnetic force in the air are at an angle of 180°–45° with the surface, while those in the iron are nearly parallel to the surface, the tangent of the angle with the surface being $\frac{1}{\mu_1}$.

Case 2.—*Electrical Waves.*

This case, in which the electrical properties are the principal factors and the electric currents and displacement are in the plane containing the normal to the surface and the direction of propagation, is obtained by making $B = 0$ in the equations. This gives the surface conditions

$$\frac{A}{\mu}(a^2 + e^2) = \frac{A_1}{\mu_1}(a^2 + e_1^2),$$
$$Ae = A_1 e_1$$

and the conditions of wave propagation

$$a^2 + e^2 = \mu c(4\pi C + \varkappa c), \quad a^2 + e_1^2 = \mu_1 c(4\pi C_1 + \varkappa_1 c).$$

These give

$$a^2 = c\,(4\pi C + \kappa c)(4\pi C_1 + \kappa_1 c)\,\frac{\mu_1(4\pi C + \kappa c) - \mu\,(4\pi C_1 + \kappa_1 c)}{(4\pi C + \kappa c)^2 - (4\pi C_1 + \kappa_1 c)^2}\,,$$

$$e^2 = c\,(4\pi C + \kappa c)^2\,\frac{\mu\,(4\pi C + \kappa c) - \mu_1(4\pi C_1 + \kappa_1 c)}{(4\pi C + \kappa c)^2 - (4\pi C_1 + \kappa_1 c)^2}\,,$$

$$e_1^2 = c\,(4\pi C_1 + \kappa_1 c)^2\,\frac{\mu\,(4\pi C + \kappa c) - \mu_1(4\pi C_1 + \kappa_1 c)}{(4\pi C + \kappa c)^2 - (4\pi C_1 + \kappa_1 c)^2}\,.$$

The separation of the real and imaginary parts causes too complicated results, and therefore it is better to simplify them before reduction by application to the case of air and a metal. For this, $C = 0$, $\mu = 1$, $\kappa_1 = 0$ or is at least small compared with C_1. This gives

$$\alpha = -\frac{b^2\kappa^2\mu_1}{8\pi C_1} = 0 \text{ nearly},$$

$$\beta = b\sqrt{\kappa}\left\{1 + \frac{1}{8}\,\frac{\kappa^2 b^2\mu_1^2}{(4\pi C_1)^2}\right\} = b\sqrt{\kappa} \text{ nearly},$$

$$\gamma = \delta = -\kappa\sqrt{\frac{b^3\mu_1}{4\pi C_1}}\,,$$

$$\gamma_1 = \delta_1 = \sqrt{2\pi C_1 b\mu_1}.$$

The waves thus proceed with very little damping, and have the velocity of light $\sqrt{\frac{1}{\kappa}}$, very nearly. In each of the media, as we pass away perpendicular to the surfaces, the waves are of the diffusion type. Putting

$$\phi = \beta x + \delta y - bt,$$
$$\phi_1 = \beta x + \delta_1 y - bt,$$

the disturbance is

Vector Potential.

In the Air.	In the Metal.
$F = A\gamma\varepsilon^{\alpha x + \gamma y}(\cos\phi - \sin\phi)$,	$F_1 = A\gamma\varepsilon^{\alpha x + \gamma_1 y}(\cos\phi_1 - \sin\phi_1)$,
$G = -A\varepsilon^{\alpha x + \gamma y}(\alpha\cos\phi - \beta\sin\phi)$,	$G_1 = -A\,\frac{\kappa b}{4\pi C_1}\,\varepsilon^{\alpha x + \gamma_1 y}(\alpha\sin\phi_1 + \beta\cos\phi_1)$,
$H = 0$,	$H_1 = 0$.

Magnetic Induction.

In the Air.	In the Metal.
$\bar{a} = \bar{b} = 0,$	$\bar{a}_1 = \bar{b}_1 = 0,$
$\bar{c} = Ab^2\kappa\varepsilon^{\alpha x + \gamma y}\cos\phi.$	$\bar{c}_1 = Ab^2\kappa\mu_1\varepsilon^{\alpha x + \gamma_1 y}\cos\phi_1.$

Electric Current.

$u = \quad A\dfrac{b^2\kappa\gamma}{4\pi}\varepsilon^{\alpha x + \gamma y}(\cos\phi - \sin\phi),$	$u_1 = \quad A\dfrac{C_1 b\gamma}{\sqrt{2}}\varepsilon^{\alpha x + \gamma_1 y}(\cos\phi_1 - \sin\phi_1),$
$v = -A\dfrac{b^2\kappa}{4\pi}\varepsilon^{\alpha x + \gamma y}(\alpha\cos\phi - \beta\sin\phi),$	$v_1 = -A\dfrac{b^2\kappa}{4\pi}\varepsilon^{\alpha x + \gamma_1 y}(\alpha\cos\phi_1 - \beta\sin\phi_1),$
$w = 0.$	$w_1 = 0.$

Surface density of electricity is

$$\sigma = -A\frac{b\kappa}{4\pi}\varepsilon^{\alpha x}\{\beta\cos(\beta x - bt) + \alpha\sin(\beta x - bt)\}.$$

As the electricity is not at rest, there is no such thing as a potential.

The case we have solved is that of waves of electromagnetic disturbance advancing along the surface with a velocity a little less than that of light, with only a very small damping factor, accompanied with an advancing electrostatic charge upon the surface. The waves die out as one goes away from the surface, according to the very small factor $\varepsilon^{-\frac{b^2\kappa^2\mu_1}{8\pi C_1}y}$ in the air and to the large factor $\varepsilon^{-\sqrt{2\pi C_1 b\mu_1}\,y}$ in the metal. Indeed in the latter the waves are of the diffusion type which die away to $\frac{1}{535}$th part in a complete wave as discussed in the case of magnetic waves. The wave length perpendicular to the surface is $\dfrac{\sqrt{2}}{\sqrt{C_1\nu_1\mu_1}}$, the same as for the magnetic waves. For $\nu_1 = 200$ reversals per second, this is .32 cm. for iron and 4 cm. for copper. In other words, the current is diffused downward into the copper more than twelve times as fast as into iron. In copper, at a depth of 5 mm., the current is .45 at the surface, with 200 reversals per second; in iron the distance is only 0.4 mm. or about $\frac{1}{60}$ inch. These results are for a plane only, and in case of wires the current is diffused inward from all directions, and so much more than one-half the current reaches the center of a wire 10 mm. in diameter. I shall treat this case later.

The waves in both the air and the metal are plane waves whose intensity diminishes as one passes away from the surface. In the air, the wave surface is nearly perpendicular to the metal surface, while in the metal it is almost parallel to it. The equation to these wave surfaces is $\beta x + \delta y - bt =$ constant and $\beta x + \delta_1 y - bt =$ constant.

Two Metal Strips, Electrical Waves.

Let us now consider the case of two metal strips of infinite extent connected along one edge to the poles of an alternating dynamo. For a practical case, we can limit the strips to a certain width, provided their distance apart is small in proportion to their width. Let the strips be of the same material and the same thickness. There will then be a central plane of symmetry at which the condition is that the displacement currents and electromotive forces along it shall be zero and the magnetic force and induction perpendicular to it shall be zero.

Let the plane $y = 0$ be this plane, and the strip on the positive side be between $y = D''$ and $y = D'$. Then the vector potential in the three spaces can be written

$$\text{Central space,} \left[\begin{aligned} F &= Ae\{\varepsilon^{ey} - \varepsilon^{-ey}\}\,\varepsilon^{ax+ct}, \\ G &= - Aa\{\varepsilon^{ey} + \varepsilon^{-ey}\}\,\varepsilon^{ax+ct}, \end{aligned}\right.$$

$$\text{Metal strips,} \left[\begin{aligned} F' &= e'\{A'\varepsilon^{e'y} - B'\varepsilon^{-e'y}\}\,\varepsilon^{ax+ct}, \\ G' &= - a\{A'\varepsilon^{e'y} + B'\varepsilon^{-e'y}\}\,\varepsilon^{ax+ct}, \end{aligned}\right.$$

$$\text{Outside space,} \left[\begin{aligned} F'' &= e''A''\varepsilon^{e''y+ax+ct}, \\ G'' &= - aA''\varepsilon^{e''y+ax+ct}. \end{aligned}\right.$$

These satisfy the conditions at the central plane and also the equation of continuity. For wave propagation we also have

$$a^2 = -e^2 + c^2\varkappa\mu = -e'^2 + 4\pi C'c\mu' = -e''^2 + c^2\varkappa\mu,$$

whence

$$e''^2 = e^2.$$

The conditions at the surfaces of the metal are

$$c^2\varkappa A''\varepsilon^{e''D''} = 4\pi C'c\{A'\varepsilon^{e'D''} + B'\varepsilon^{-e'D''}\} \mid e''A''\varepsilon^{e''D''} = e'\{A'\varepsilon^{e'D''} - B'\varepsilon^{-e'D''}\},$$

$$4\pi C'c\{A'\varepsilon^{e'D'} + B'\varepsilon^{-e'D'}\} = c^2\varkappa A\{\varepsilon^{eD'} + \varepsilon^{-eD'}\} \mid e'\{A'\varepsilon^{e'D'} - B'\varepsilon^{-e'D'}\} = eA\{\varepsilon^{eD'} - \varepsilon^{-eD'}\}.$$

These, with the three above, give 7 equations for determining a, e, e', e'' and A'', A' and B' in terms of A.

Putting $E = \varkappa\sqrt{\frac{c^3\mu'}{4\pi C'}}\,\frac{\varepsilon^{2e'(D''-D')}+1}{\varepsilon^{2e'(D''-D')}-1}$ and noting that eD' is very small, we have the equation

$$e^3 - Ee^2 + \frac{E}{D'}\,e - \frac{c^3\varkappa^2\mu'}{4\pi C'D'} = 0.$$

The first, and very near, approximation to the value of e is

$$e^2 = -\frac{E}{D'} = -\frac{\varkappa}{D'}\sqrt{\frac{c^3\mu'}{4\pi C'}}\,\frac{\varepsilon^{2e'(D''-D')}+1}{\varepsilon^{2e'(D''-D')}-1},$$

in which $e' = \sqrt{4\pi C'c\mu'}$ very nearly.

This gives $a^2 = c^2\varkappa\mu\left\{1 + \frac{1}{\mu D'}\sqrt{\frac{\mu'}{4\pi C'c}}\,\frac{\varepsilon^{2e'(D''-D')}+1}{\varepsilon^{2e'(D''-D')}-1}\right\}$. When the thickness of the strips, $D'' - D'$, is small, we have, since $\varkappa\mu = \frac{1}{v^2}$ and $\mu = 1$ nearly, for all insulators,

$$a^2 = (\alpha + i\beta)^2 = -\frac{b^2}{v^2} - \frac{ib}{4\pi C'D'(D''-D')v^2}.$$

Hence

$$\alpha^2 = \frac{b^2}{2v^2}\left\{\sqrt{\frac{1}{(4\pi C'D'(D''-D')b)^2} + 1} - 1\right\},$$

$$\beta^2 = \frac{b^2}{2v^2}\left\{\sqrt{\frac{1}{(4\pi C'D'(D''-D')b)^2} + 1} + 1\right\}.$$

The limiting case, when $C'D'(D''-D')$ is small, is

$$\beta = -\alpha = \frac{1}{v}\sqrt{\frac{b}{8\pi C'D'(D''-D')}}.$$

The waves are then of the diffusion type, giving a velocity of

$$\frac{b^2}{\beta} = v\sqrt{8\pi C'D'(D''-D')\,b},$$

which is the same as the solution of Sir William Thomson for a cable whose capacity and resistance per unit of length are $\frac{\varkappa}{4\pi D'}$ and $\frac{1}{C'(D''-D')}$ respectively.

But when the thickness of the strips increases in value, the damping factor α decreases and the velocity increases until we reach the limit for which

$$a^2 = -\frac{b^2}{v^2}\left\{1 + \frac{1}{\mu D'}\sqrt{\frac{\mu'}{8\pi C'b}}\,(1+i)\right\}$$

which gives

$$\alpha^2 = \frac{b^2}{2v^2}\left\{\sqrt{1 + \frac{\mu'}{4\pi C'b\mu^2 D'^2} + \frac{2}{\mu D'}\sqrt{\frac{\mu'}{8\pi C'b}}} - 1 - \frac{1}{\mu D'}\sqrt{\frac{\mu'}{8\pi C'b}}\right\},$$

$$\beta^2 = \frac{b^2}{2v^2}\left\{\sqrt{1 + \frac{\mu'}{4\pi C'b\mu^2 D'^2} + \frac{2}{\mu D'}\sqrt{\frac{\mu'}{8\pi C'b}}} + 1 + \frac{1}{\mu D'}\sqrt{\frac{\mu'}{8\pi C'b}}\right\}.$$

When the conductivity is large, the damping factor vanishes and the velocity of the wave is that of light, as it should be.

The Expression of any Differential Coefficient of a Function of a Function of any number of Variables by aid of the corresponding Differential Coefficients of any n Powers of the Function, where n is the Order of the Differential Coefficient.

By J. C. Fields.

Let $x, y, z \dots .$ be any number of independent variables, u any function of these variables, and Δ any function of the degree n of the symbols $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz} \dots,$ n being a positive integer. Evidently

$$\left(\frac{d}{dx}\right)^n u^m = g(m).u^m; \left(\frac{d}{dx}\right)^r \left(\frac{d}{dy}\right)^s \dots . u^m = h(m).u^m,$$

where $g(m)$, $h(m)$, considered with regard to m alone, are polynomials in this quantity of the degrees n and $r+s+\dots .$ respectively, whence plainly $\Delta u^m = f(m)\, u^m$ where $f(m)$ is a polynomial of the degree n in m. We have therefore, by the ordinary theory of partial fractions,

$$1. \quad \Delta u^m = f(m)\, u^m = \theta(m).\frac{f(m)}{\theta(m)} u^m = \theta(m)\, u^m \sum_0^p {}_r \frac{f(\alpha_r)}{\theta'(\alpha_r)} \cdot \frac{1}{m-\alpha_r}$$

$$= \theta(m) \sum_0^p {}_r \frac{u^{m-\alpha_r}}{m-\alpha_r} \cdot \frac{f(\alpha_r)\, u^{\alpha_r}}{\theta'(\alpha_r)} = \theta(m) \sum_0^p {}_r \frac{u^{m-\alpha_r}}{m-\alpha_r} \cdot \frac{\Delta u^{\alpha_r}}{\theta'(\alpha_r)},$$

where $\theta(m) = (m-\alpha_0)(m-\alpha_1) \dots . (m-\alpha_p)$, the α's being any constant quantities all distinct from one another and greater than n in number. In particular, if we put $p=n$, $\alpha_r = r$ for all values of r, $\Delta \equiv \left(\frac{d}{dx}\right)^\rho \left(\frac{d}{dy}\right)^\sigma \left(\frac{d}{dz}\right)^\tau \dots .$ where $\rho+\sigma+\tau+\dots . = n$, and designate by $B_{m,\rho,\sigma \dots .}$ the coefficient of $x^\rho y^\sigma z^\tau \dots .$ in the development of u^m, we have on dividing equation (1) through by $\rho!\,\sigma!\,\tau!\ldots.$, putting $x=0$, $y=0$, $z=0\ldots.$, and noticing that $\theta'(r)=(-1)^{n-r}r!\,(n-r)!$

$$2.\qquad B_{m,\rho,\sigma\ldots.}=\theta(m)\sum_{0}^{n}{}_r\frac{u_0^{m-r}}{m-r}\cdot\frac{B_{r,\rho,\sigma\ldots.}}{\theta'(r)}$$

$$=m(m-1)\ldots.(m-n)\sum_{1}^{n}{}_r\frac{u_0^{m-r}}{m-r}\cdot\frac{(-1)^{n-r}B_{r,\rho,\sigma,\ldots.}}{r!\,(n-r)!},$$

where u_0 expresses the value of u when $x=0$, $y=0$, $z=0\ldots.$

In particular, if u be a function of one variable (x) only, and $\Delta\equiv\left(\frac{d}{dx}\right)^n$,

$$B_{m,n}=m(m-1)\ldots.(m-n)\sum_{1}^{n}{}_r\frac{u_0^{m-r}}{m-r}\cdot\frac{(-1)^{n-r}B_{r,n}}{r!\,(n-r)!},$$

a formula originally due to a suggestion of Eisenstein, and of which a proof has been given by Sylvester.*

We might notice more generally that $\Delta u^m v=F(m)u^m$ where u and v are any functions of the variables and $F(m)$ is a polynomial of the n^{th} degree in m, whence, in the same way as (1), we derive

$$3.\qquad \Delta u^m v=\theta(m)\sum_{0}^{p}{}_r\frac{u^{m-a_r}}{m-a_r}\cdot\frac{\Delta u^{a_r}v}{\theta'(a_r)}.$$

Hence we see that formula (2) holds more generally where $B_{m,\rho,\sigma\ldots.}$ is the coefficient of $x^\rho y^\sigma\ldots.$ in $u^m v$.

If we have $\phi(u)=\sum a_m u^m$ (the a's being constant coefficients), and apply (1), we get

$$4.\qquad \Delta\phi(u)=\sum_{0}^{p}{}_r\frac{u^{-a_r}\Delta u^{a_r}}{\theta'(a_r)}\left(\sum{}_m\frac{\theta(m)}{m-a_r}a_m u^m\right)=\sum_{0}^{p}{}_r\frac{u^{-a_r}\Delta u^{a_r}}{\theta'(a_r)}\cdot\frac{\theta\left(u\frac{d}{du}\right)}{\frac{d}{du}-a_r}\phi(u).$$

If, in (4), we put $p=n$, and $a_r=r$ for all values of r, then

$$\theta\left(u\frac{d}{du}\right)=u\frac{d}{du}\left(u\frac{d}{du}-1\right)\left(u\frac{d}{du}-2\right)\ldots.\left(u\frac{d}{du}-n\right)=u^{n+1}\left(\frac{d}{du}\right)^{n+1}$$

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* Quarterly Journ. of Math., Vol. I, p. 199. Also Bertrand, Cal. Diff., p. 181.

and we have

5. $$\Delta\phi(u) = \sum_{r=0}^{n} \frac{(-1)^{n-r}u^{-r}\Delta u^r}{r!\,(n-r)!} \cdot \frac{u^{n+1}\left(\frac{d}{du}\right)^{n+1}}{u\frac{d}{du} - r}\phi(u)$$

$$= u^{n+1}\sum_{r=0}^{n} \frac{(-1)^{n-r}u^{-r}\Delta u^r}{r!\,(n-r)!}\left(\frac{d}{du}\right)^{n+1} . u^r \int u^{-(r+1)}\phi(u)\,du$$

$$= u^{n+1}\sum_{r=0}^{n}\sum_{s=0}^{r} \frac{(-1)^{n-r}u^{-r}\Delta u^r}{(n-r)!\,(r-s)!}\left(\frac{d}{du}\right)^{n-s}\frac{\phi(u)}{u^{s+1}},$$

or if in (5) we operate first with $u^{n+1}\left(\frac{d}{du}\right)^{n+1}$ and afterwards with $\left(u\frac{d}{du} - r\right)^{-1}$, we get

6. $$\Delta\phi(u) = \sum_{r=0}^{n} \frac{(-1)^{n-r}\Delta u^r}{r!\,(n-r)!}\int u^{n-r}\phi^{(n+1)}(u)\,du,$$

where, in performing the integration by successive partial integrations, we put zero for the arbitrary constant, or what is the same thing, the integral is the sum of a number of terms each one of which contains as a factor $\phi(u)$ or some differential coefficient of $\phi(u)$, as is readily seen from (5), where the whole operation upon $\phi(u)$ is direct and not inverse; thus from (6) we obtain

7. $$\Delta\phi(u) = \sum_{\rho=0}^{n}\sum_{r=0}^{\rho} \frac{(-1)^{\rho-r}u^{\rho-r}\Delta u^r}{r!\,(\rho-r)!}\phi^\rho(u) = \sum_{r=0}^{n}\sum_{\rho=r}^{n} \frac{(-1)^{\rho-r}u^{\rho-r}\phi^\rho(u)}{r!\,(\rho-r)!} . \Delta u^r.$$

For example, we find

$$\Delta e^u = e^u \sum_{r=0}^{n}\sum_{\rho=r}^{n} \frac{(-1)^{\rho-r}u^{\rho-r}\Delta u^r}{r!\,(\rho-r)!},$$

$$\Delta \log u = -\sum_{r=0}^{n}\sum_{\rho=r}^{n} \frac{1}{\rho}\binom{\rho}{r}(-1)^r u^{-r}\Delta u^r.$$

We see that $\sum_{r=0}^{\rho} \frac{(-1)^{\rho-r}\rho!}{r!\,(\rho-r)!}u^{-r}\Delta u^r = \Delta\left(\frac{u}{\alpha} - 1\right)^\rho$ where, when the operation Δ has been performed (α being treated as a constant), α is replaced by u. We have therefore from (7),

8. $$\Delta\phi(u) = \sum_{\rho=0}^{n} \frac{u^\rho\phi^\rho(u)}{\rho!}\Delta\left(\frac{u}{\alpha} - 1\right)^\rho,$$

where, after performance of operation Δ, α is replaced by u.

In particular, when u is a function of one variable (x) only, and we put $\Delta \equiv \left(\frac{d}{dx}\right)^n$ in (8), we obtain $\left(\frac{d}{dx}\right)^n \phi(u) = \sum_1^n {}_\rho \frac{u^\rho \phi^\rho(u)}{\rho!} \left(\frac{d}{dx}\right)^n \left(\frac{u}{\alpha} - 1\right)^\rho$, a symbolic expression for $\left(\frac{d}{dx}\right)^n \phi(u)$ given by Bertrand.*

Again we have $\varphi\left(u\frac{d}{du}\right) u^n = \varphi(n)\, u^n = E_y^n \varphi(y)\, u^y$, $(y = 0)$, whence $\varphi\left(u\frac{d}{du}\right) \sum a_n u^n = \sum a_n E_y^n \varphi(y)\, u^y$ $(y = 0)$ and

$$9. \qquad \varphi\left(u\frac{d}{du}\right) \chi(u) = \chi(E_y)\, \varphi(y)\, u^y, \quad (y = 0).$$

We might notice in passing that putting $u = e^x$, $\chi(u) = \psi(x)$, $\chi(E_y) = \psi\left(\frac{d}{dy}\right)$ we obtain the theorem

$$10. \qquad \varphi\left(\frac{d}{dx}\right) \psi(x) = \psi\left(\frac{d}{dy}\right) \cdot \varphi(y)\, e^{xy} = \psi\left(x + \frac{d}{dy}\right) \varphi(y), \quad (y = 0).$$

If in (4) we put $\phi(u) = (\log u)^s$, we get, with the help of (9),

$$\Delta (\log u)^s = \sum_0^p {}_r \frac{u^{-a_r} \Delta u^{a_r}}{\theta'(a_r)} \frac{\theta\left(u\frac{d}{du}\right)}{u\frac{d}{du} - a_r} (\log u)^s = \sum_0^p {}_r \frac{u^{-a_r} \Delta u^{a_r}}{\theta'(a_r)} \cdot (\log E_y)^s \frac{\theta(y)\, u^y}{y - a_r}$$

$$= \sum_0^p {}_r \frac{u^{-a_r} \Delta u^{a_r}}{\theta'(a_r)} \cdot \left(\frac{d}{dy}\right)^s \frac{\theta(y)\, u^y}{y - a_r}$$

$$= \sum_0^p {}_r \frac{u^{-a_r} \Delta u^{a_r}}{\theta'(a_r)} u^y \left(\log u + \frac{d}{dy}\right)^s \frac{\theta(y)}{y - a_r}, \quad (y = 0),$$

thus,

$$11. \qquad \Delta (\log u)^s = \left(\log u + \frac{d}{dm}\right)^s \sum_0^p {}_r \frac{u^{-a_r} \Delta u^{a_r}}{\theta'(a_r)} \cdot \frac{\theta(m)}{m - a_r}, \quad (m = 0).$$

We might also obtain this result directly from (1) by differentiating with regard to m, thus

$$\Delta (\log u)^s u^m = \left(\frac{d}{dm}\right)^s \Delta u^m = \left(\frac{d}{dm}\right)^s . \, u^m (u^{-m} \Delta u^m) = u^m \left(\log u + \frac{d}{dm}\right)^s u^{-m} \Delta u^m;$$

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* Cal. Diff., p. 141.

substituting in this formula for Δu^m from (1), we obtain

$$\Delta(\log u)^s u^m = u^m\left(\log u + \frac{d}{dm}\right)^s \sum_{0}^{p}{}_r \frac{u^{-\alpha_r}\Delta u^{\alpha_r}}{\theta'(\alpha_r)}\;\frac{\theta(m)}{m-\alpha_r},$$

and on putting $m=0$ in this formula we obtain (11).

Supposing $(\log u)^s$ to be developed in powers of $x, y, z \ldots .$, put in particular $u_0 = 1$ (therefore $\log u_0 = 0$),

$$\Delta \equiv \left(\frac{d}{dx}\right)^{\rho}\left(\frac{d}{dy}\right)^{\sigma}\left(\frac{d}{dz}\right)^{\tau}\ldots ., \qquad p = n = \rho + \sigma + \tau + \ldots .;$$

also designate by $A_{s,\rho,\sigma,\tau\ldots .}$ the coefficient of $x^{\rho}y^{\sigma}z^{\tau}\ldots .$ in the development of $(\log u)^s$, then (11) gives

$$A_{s,\rho,\sigma,\tau\ldots .} = \frac{1}{\rho!\,\sigma!\,\tau!\ldots .}\left(\frac{d}{dx}\right)^{\rho}\left(\frac{d}{dy}\right)^{\sigma}\underset{0=x=y=z=\ldots .}{\left(\frac{d}{dz}\right)^{\tau}}\ldots .(\log u)^s$$

$$= \frac{1}{\rho!\,\sigma!\ldots .}\left(\log u_0 + \frac{d}{dm}\right)^s \underset{m=0=x=y=\ldots .}{\sum_{0}^{n}{}_r} \frac{u_0^{-\alpha_r}\Delta u^{\alpha_r}}{\theta'(\alpha_r)}\;\frac{\theta(m)}{m-\alpha_r}.$$

We have therefore

$$12.\qquad A_{s,\rho,\sigma,\tau\ldots .} = \frac{1}{\rho!\,\sigma!\ldots .}\left(\frac{d}{dm}\right)^s \underset{m=0=x=y=\ldots .}{\sum_{0}^{n}{}_r} \frac{\Delta u^{\alpha_r}}{\theta'(\alpha_r)}\cdot\frac{\theta(m)}{m-\alpha_r}$$

$$= \left(\frac{d}{dm}\right)^s \sum_{1}^{n}{}_r \frac{B_{\alpha_r,\rho,\sigma,\ldots .}}{\theta'(\alpha_r)}\cdot\frac{\theta(m)}{m-\alpha_r}$$

$$= \left(\frac{d}{dm}\right)^s \sum_{1}^{n}{}_r \frac{(-1)^{n-r}B_{r,\rho,\sigma\ldots .}}{r!\,(n-r)!}\cdot\frac{\theta(m)}{m-r}, \quad (m=0), \quad \theta(m) = m(m-1)\ldots(m-n).$$

In the same manner that we obtained (4) from (1), from (11) we obtain

$$13.\qquad \Delta\Phi(\log u) = \Phi\left(\log u + \frac{d}{dm}\right)\sum_{0}^{p}{}_r \frac{u^{-\alpha_r}\Delta u^{\alpha_r}}{\theta'(\alpha_r)}\cdot\frac{\theta(m)}{m-\alpha_r}, \quad (m=0).$$

Putting $u = \chi(v)$, from (4) we get

$$\Delta u^{\alpha_r} = \Delta\{\chi(v)\}^{\alpha_r} = \sum_{0}^{p}{}_s \frac{v^{-\beta_s}\Delta v^{\beta_s}}{\theta_1'(\beta_s)}\cdot\frac{\theta_1\left(v\dfrac{d}{dv}\right)}{v\dfrac{d}{dv}-\beta_s}\{\chi(v)\}^{\alpha_r}.$$

Substituting in (4),

$$\Delta\Phi\chi(v) = \sum_{0}^{p}{}_r \sum_{0}^{p}{}_s \frac{u^{-\alpha_r}\theta_2\left(u\dfrac{d}{du}\right)}{\theta_2'(\alpha_r)\left(u\dfrac{d}{du}-\alpha_r\right)}\Phi(u)\cdot\frac{v^{-\beta_s}\Delta v^{\beta_s}}{\theta_1'(\beta_s)}\cdot\frac{\theta_1\left(v\dfrac{d}{dv}\right)}{v\dfrac{d}{dv}-\beta_s}\{\chi(v)\}^{\alpha_r},$$

where

$\theta_1(m) = (m-\beta_0)(m-\beta_1)\dots(m-\beta_p)$, $\theta_2(m) = (m-\alpha_0)(m-\alpha_1)\dots(m-\alpha_p)$.

By successive applications of (4) we will get in general

$$14.\quad \Delta\Phi_\kappa\Phi_{\kappa-1}\dots\Phi_1(u) = \sum_0^p{}_q \sum_0^p{}_r \dots \sum_0^p{}_s \sum_0^p{}_t \sum_0^p{}_w \frac{\Phi_{\kappa-1}^{-\alpha_{\kappa,q}}\theta_\kappa\left(\Phi_{\kappa-1}\frac{d}{d\Phi_{\kappa-1}}\right)}{\theta'_\kappa(\alpha_{\kappa,q})\left(\Phi_{\kappa-1}\frac{d}{d\Phi_{\kappa-1}} - \alpha_{\kappa,q}\right)}\Phi_\kappa$$

$$\times \frac{\Phi_{\kappa-2}^{-\alpha_{\kappa-1,r}}\theta_{\kappa-1}\left(\Phi_{\kappa-2}\frac{d}{d\Phi_{\kappa-2}}\right)}{\theta'_{\kappa-1}(\alpha_{\kappa-1,r})\left(\Phi_{\kappa-2}\frac{d}{d\Phi_{\kappa-2}} - \alpha_{\kappa-1,r}\right)}\Phi_{\kappa-1}^{\alpha_{\kappa,q}}\dots\frac{\Phi_1^{-\alpha_{2,t}}\theta_2\left(\Phi_1\frac{d}{d\Phi_1}\right)}{\theta'_2(\alpha_{2,t})\left(\Phi_1\frac{d}{d\Phi_1} - \alpha_{2,t}\right)}\Phi_2^{\alpha_{3,s}}$$

$$\times \frac{u^{-\alpha_{1,w}}\theta_1\left(u\frac{d}{du}\right)}{\theta'_1(\alpha_{1,w})\left(u\frac{d}{du} - \alpha_{1,w}\right)}\Phi_1^{\alpha_{2,t}}.\,\Delta u^{\alpha_{1,w}},$$

where $\theta_\nu(m) \equiv (m-\alpha_{\nu,0})(m-\alpha_{\nu,1})\dots\dots(m-\alpha_{\nu,p})$, the α_ν's being any p *distinct* quantities taken at will, of which any number may happen to coincide with α_μ's in any other function $\theta_\mu(m) \equiv (m-\alpha_{\mu,0})(m-\alpha_{\mu,1})\dots\dots(m-\alpha_{\mu,p})$; putting for all values of ν, $\theta_\nu(m) = m(m-1)\dots\dots(m-p) \equiv \theta(m)$, we have

$$15.\quad \Delta\Phi_\kappa\Phi_{\kappa-1}\dots\dots\Phi_1(u) = \sum_0^p{}_{q,r,\dots\dots s,t,w} \frac{\Phi_{\kappa-1}^{-q}\theta\left(\Phi_{\kappa-1}\frac{d}{d\Phi_{\kappa-1}}\right)}{\theta'(q)\left(\Phi_{\kappa-1}\frac{d}{d\Phi_{\kappa-1}} - q\right)}\Phi_\kappa$$

$$\times \frac{\Phi_{\kappa-2}^{-r}\theta\left(\Phi_{\kappa-2}\frac{d}{d\Phi_{\kappa-2}}\right)}{\theta'(r)\left(\Phi_{\kappa-2}\frac{d}{d\Phi_{\kappa-2}} - r\right)}\Phi_{\kappa-1}^q\dots\dots\frac{\Phi_1^{-t}\theta\left(\Phi_1\frac{d}{d\Phi_1}\right)}{\theta'(t)\left(\Phi_1\frac{d}{d\Phi_1} - t\right)}\Phi_2^s$$

$$\times \frac{u^{-w}\theta\left(u\frac{d}{du}\right)}{\theta'(w)\left(u\frac{d}{du} - w\right)}\Phi_1^t.\,\Delta u^w.$$

For brevity I have put $\Phi_1 \equiv \Phi_1(u)$, $\Phi_2 \equiv \Phi_2(\Phi_1) \equiv \Phi_2\{\Phi_1(u)\}$, etc., $\Phi_\kappa \equiv \Phi_\kappa(\Phi_{\kappa-1})$. Putting in (15)

$$\frac{\theta\left(\Phi_{\kappa-2}\frac{d}{d\Phi_{\kappa-2}}\right)}{\Phi_{\kappa-2}\frac{d}{d\Phi_{\kappa-2}} - r}\Phi_{\kappa-1}^q = \frac{\Phi_{\kappa-2}^{p+1}\left(\frac{d}{d\Phi_{\kappa-2}}\right)^{p+1}}{\Phi_{\kappa-2}\frac{d}{d\Phi_{\kappa-2}} - r}\Phi_{\kappa-1}^q$$

$$= \Phi_{\kappa-2}^r\int\Phi_{\kappa-2}^{p-r}\left(\frac{d}{d\Phi_{\kappa-2}}\right)^{p+1}\Phi_{\kappa-1}^q d\Phi_{\kappa-2},$$

where, as in (6), the integration is supposed to be performed by partial integrals, the arbitrary constant being put equal to 0, and

$$\frac{1}{\theta'(r)}=\frac{(-1)^{p-r}}{r!\,(p-r)!}\equiv\frac{(-1)^{p-r}}{p!}\binom{p}{r},$$

with similar substitutions with regard to q, s, t, etc., we get for (15)

$$16.\quad \Delta\Phi_\kappa\Phi_{\kappa-1}\ldots\ldots\Phi_1(u)=\frac{(-1)^{\kappa p}}{(p!)^\kappa}\sum_{0}^{p}{}_{q,\,r,\,\ldots\ldots t,\,w,}\,(-1)^{q+r+\ldots\ldots+t+w}\binom{p}{q}\binom{p}{r}\ldots\ldots$$

$$\ldots\ldots\binom{p}{t}\binom{p}{w}\int\Phi_{\kappa-1}^{p-q}\left(\frac{d}{d\Phi_{\kappa-1}}\right)^{p+1}\Phi_\kappa d\Phi_{\kappa-1}\int\Phi_{\kappa-2}^{p-r}\left(\frac{d}{d\Phi_{\kappa-2}}\right)^{p+1}\Phi_{\kappa-1}^{q}d\Phi_{\kappa-2}\ldots\ldots$$

$$\ldots\ldots\int\Phi_1^{p-t}\left(\frac{d}{d\Phi_1}\right)^{p+1}\Phi_2^{s}d\Phi_1\int u^{p-w}\left(\frac{d}{du}\right)^{p+1}\Phi_1^{t}du.\Delta u^{w}.$$

For example, if we have what I might call the κ-storied function $e^{e^{\cdot^{\cdot^{\cdot^{e^u}}}}}$, where e occurs κ times, and which I will designate by u_κ, from (16) we obtain

$$17.\quad \Delta u_\kappa=\frac{(-1)^{\kappa p}}{(p!)^\kappa}\sum_{0}^{p}{}_{q,\,r,\,\ldots\ldots t,\,w}\,(-1)^{q+r+\ldots\ldots+t+w}(qr\ldots st)^{p+1}\binom{p}{q}\binom{p}{r}\ldots\ldots$$

$$\ldots\ldots\binom{p}{t}\binom{p}{w}\int u_{\kappa-1}^{p-q}e^{u_{\kappa-1}}du_{\kappa-1}\int u_{\kappa-2}^{p-r}e^{qu_{\kappa-2}}du_{\kappa-2}\ldots\ldots$$

$$\ldots\ldots\int u_1^{p-t}e^{su_1}du_1\int u^{p-w}e^{tu}du.\Delta u^{w}.$$

Where each integration is supposed to be performed by partial integrals, the arbitrary constant being put equal to 0, or in other words, that particular integral being selected which explicitly contains as a factor the exponential which appears under the integral sign in question.

We might almost indefinitely generalize the above formulae, thus, instead of u^m in (1) we might suppose Φ a function of any number of variables x, y, $z\ldots$ and of any number of parameters m_1, m_2, $m_3\ldots\ldots$, and Δ any operator independent of these parameters, and such that

$$\Delta\Phi(x,\,y\ldots\ldots m_1,\,m_2\ldots\ldots)=f(m_1,\,m_2,\ldots\ldots m_\kappa\ldots\ldots).\Phi(x,\,y\ldots\ldots m_1,\,m_2\ldots\ldots),$$

where as regards the parameters alone f is of a finite degree. Supposing f to be of degrees n_1, n_2, etc. in m_1, m_2, etc. respectively, and taking any arbitrary polynomials in m's such as $\theta_\kappa(m_\kappa)=\Pi(m_\kappa-a_{\kappa,t})$ where degree of polynomial is

$\not< n_\kappa$, by successive application of the theory of partial fractions as regards m_1, m_2, etc. to f, we may express it as a summation in the form

$$f(m_1, m_2 \dots\dots m_\kappa \dots\dots)$$
$$= \theta_1(m_1)\,\theta_2(m_2)\dots\dots \sum \frac{f(\alpha_{1,r}, \alpha_{2,s} \dots\dots)}{\theta_1'(\alpha_{1,r})\,\theta_2'(\alpha_{2,s})\dots\dots} \cdot \frac{1}{(m_1-\alpha_{1,r})(m_2-\alpha_{2,s})\dots\dots}$$

whence

$$\Delta\Phi(x, y \dots\dots m_1, m_2 \dots\dots) = f(m_1, m_2 \dots\dots)\,\Phi(x, y \dots\dots m_1, m_2 \dots\dots)$$
$$= \Pi\theta_\kappa(m_\kappa) \sum_{r,s\dots\dots} \frac{f(\alpha_{1,r}, \alpha_{2,s} \dots)\,\Phi(x, y \dots\dots \alpha_{1,r}, \alpha_{2,s}, \dots\dots)}{\theta_1'(\alpha_{1,r})\,\theta_2'(\alpha_{2,s})\dots\dots}$$
$$\times \frac{1}{(m_1-\alpha_{1,r})(m_2-\alpha_{2,s})\dots\dots} \cdot \frac{\Phi(x, y \dots\dots m_1, m_2 \dots\dots)}{\Phi(x, y \dots\dots \alpha_{1,r}, \alpha_{2,s} \dots\dots)}$$
$$= \Pi\theta_\kappa(m_\kappa) \sum_{r,s\dots\dots} \frac{\Delta\Phi(x, y \dots\dots \alpha_{1,r}, \alpha_{2,s} \dots\dots)}{\theta_1'(\alpha_{1,r})\,\theta_2'(\alpha_{2,s})\dots\dots}$$
$$\times \frac{1}{(m_1-\alpha_{1,r})(m_2-\alpha_{2,s})\dots\dots} \cdot \frac{\Phi(x, y \dots\dots m_1, m_2 \dots\dots)}{\Phi(x, y \dots\dots \alpha_{1,r}, \alpha_{2,s} \dots\dots)}\,.$$

Bernoulli's Numbers.

We might notice the application of (7) to Bernoulli's Numbers; in (7) put $\Delta \equiv \left(\frac{d}{dx}\right)^n$, $u = e^x$, $\Phi(u) = (1+u)^{-1}$, then $u_0 = 1$ and

$$\left(\frac{d}{dx}\right)^n (1+e^x)^{-1}_{x=0}$$
$$= \sum_{r=0}^{n} \sum_{\rho=r}^{n} \frac{(-1)^{\rho-r} u_0^{\rho-r} \Phi^\rho(u_0)}{r!\,(n-r)!} \left(\frac{d}{dx}\right)^n_{x=0} e^{rx} = \sum_{r=1}^{n} \sum_{\rho=r}^{n} \frac{(-1)^r \rho!}{r!\,(\rho-r)!} \cdot \frac{r^n}{2^{\rho+1}}$$
$$= \sum_{r=1}^{n} \sum_{\rho=r}^{n} \frac{(-1)^r r^n}{r!\,2^{r+1}} \left(\frac{d}{dx}\right)^r_{x=\frac{1}{2}} x^\rho = \sum_{r=1}^{n} \frac{(-1)^r r^n}{r!\,2^{r+1}} \left(\frac{d}{dx}\right)^r \frac{1-x^{n+1}}{1-x}_{x=\frac{1}{2}}$$
$$= \sum_{r=1}^{n} \frac{(-1)^r r^n}{r!\,2^{r+1}} \left\{ \left(\frac{d}{dx}\right)^r \frac{1}{1-x} - \sum_{s=0}^{r} \frac{r!}{s!\,(r-s)!} \left(\frac{d}{dx}\right)^s x^{n+1} . \left(\frac{d}{dx}\right)^{r-s} \frac{1}{1-x} \right\}_{x=\frac{1}{2}}$$
$$= \sum_{r=1}^{n} (-1)^r r^n \left\{ 1 - \frac{1}{2^{n+1}} \sum_{s=0}^{r} \binom{n+1}{s} \right\} = \frac{1}{2^{n+1}} \sum_{r=1}^{n} \sum_{s=r+1}^{n+1} (-1)^r r^n \binom{n+1}{s}$$
$$= \frac{1}{2^{n+1}} \sum_{s=2}^{n+1} \sum_{r=1}^{s-1} \binom{n+1}{s} (-1)^r r^n = \frac{1}{2^{n+1}} \sum_{r=1}^{n} (-1)^r r^n M_{n-r} = \frac{1}{2^{n+1}} \sum_{s=2}^{n+1} \binom{n+1}{s} N_{s-1}$$

where $$M_t \equiv 1 + \binom{n+1}{1} + \binom{n+1}{2} + \ldots . + \binom{n+1}{t}$$

and $$N_t \equiv -1^n + 2^n - 3^n + \ldots . + (-1)^t t^n.$$

We know that Bernoulli's p^{th} number $B_p = \frac{(-1)^p (2p)!}{2^{2p} - 1}$ multiplied by coefficient of x^{2p-1} in $(1 + e^x)^{-1}$, putting $n \equiv 2p - 1$, we have therefore

$$18. \qquad Bp = \frac{2p(-1)^p}{2^{2p} - 1} \left(\frac{d}{dx}\right)^{2p-1} (1 \underset{x=0}{+} e^x)^{-1}$$

$$= \frac{(-1)^p p}{2^{2p-1}(2^{2p} - 1)} \sum_{1}^{2p-1}{}_r (-1)^r r^{2p-1} M_{2p-r-1} = \frac{(-1)^p p}{2^{2p-1}(2^{2p} - 1)} \sum_{2}^{2p}{}_s \binom{2p}{s} N_{s-1}.$$

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Stieltjes, T. J., Note sur un développement de l'intégrale $\int_0^a e^{x^2}dx$.

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Tchebycheff, P., Sur la représentation des valeurs limites des intégrales par des résidus intégraux. Traduit du russe par Sophie Kowalevski.

Tchebycheff, P., Sur les sommes composées des coefficients des séries à termes positifs.

Weber, H., Theorie der Abel'schen Zahlkörper. IV. Über die Bildung Abel'scher Körper mit gegebener Gruppe.

Zeller, Ch., Kalender-Formeln.

Band X.

Bohlin, K., Über die Bedeutung des Princips der lebendigen Kraft für die Frage von der Stabilität dynamischer Systeme.

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CONTENTS.

PAGE
On the Surfaces with Plane or Spherical Curves of Curvature. By PROFESSOR CAYLEY, 293
On the Geometry of a Nodal Circular Cubic. By F. MORLEY, ... 307
On the Functions defined by Differential Equations, with an Extension of the Puiseux Polygon Construction to these Equations. By HENRY B. FINE, ... 317
Sur les solutions singulières des équations différentielles simultanées. Par E. GOURSAT, 329
Electromagnetic Waves and Oscillations at the Surface of Conductors. By HENRY A. ROWLAND, ... 373
The Expression of any Differential Coefficient of a Function of a Function of any number of Variables by aid of the corresponding Differential Coefficients of any n Powers of the Function, where n is the Order of the Differential Coefficient. By J. C. FIELDS, ... 388

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